Unit D1
Numbers

Introduction to Book D

In this book you will begin your study of the analysis units of this module. *Analysis* is the branch of mathematics that deals in a precise, quantitative way with the concept of a limit, and with the related ideas of infinite sums, continuous functions, differentiation and integration.

The setting for our study of analysis will be real functions – that is, functions whose domains and codomains are subsets of the real line $\mathbb{R}$. In this book we begin to study such functions from a precise point of view in order to prove many of their properties. You have met many of these properties before, and some may seem intuitively obvious, but this work will provide a sound basis for the study of more difficult properties of functions later in the module.

For example, consider the question:

Does the graph of $y = 2^x$ have a gap at $x = \sqrt{2}$?

The graph in Figure 1 does not appear to have any gaps but we need to check this carefully. Later in the book we answer this question properly by showing that the function $f(x) = 2^x$ has a property called *continuity*, so its graph has no gaps.

Before we can tackle this question, however, there is a preliminary question we must answer:

What precisely is meant by $2^{\sqrt{2}}$?

Thus, to answer a question about real *functions*, we first need to clarify our ideas about real *numbers*.


Figure 1   The graph of $y = 2^x$

Introduction

In this unit you will make a deeper study of the *real numbers*, which were introduced in Book A. You will extend your understanding of their properties, and see how they can be represented as infinite decimals. You will meet the rules for manipulating *inequalities*, which play a crucial role in analysis. You will learn how to solve and prove inequalities, and see proofs of several standard results that will be needed in later units. Finally, you will meet the concept of a *least upper bound* and see that the real numbers have the Least Upper Bound Property; this is of great importance in analysis.
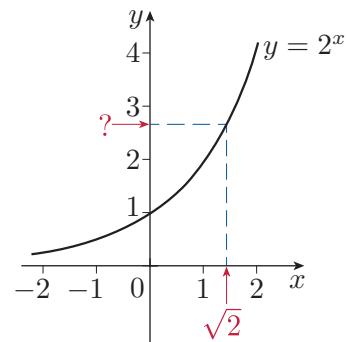
1   Real numbers

In this section we discuss the real numbers and look at some of their properties. We start by investigating the decimal representations of the rationals, and then proceed to the irrational numbers.

1.1   Rational numbers

The set of **natural numbers** is

$$\mathbb{N} = \{1, 2, 3, \ldots\},$$

the set of **integers** is

$$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$$

and the set of **rational numbers** (or **rationals**) is

$$\mathbb{Q} = \{p/q : p \in \mathbb{Z}, q \in \mathbb{N}\}.$$

(Note that we do not include 0 in the natural numbers, though some mathematical texts do.)

Remember that each rational number has many different representations as a ratio of integers; for example,

$$\frac{1}{3} = \frac{2}{6} = \frac{10}{30} = \cdots.$$

The usual arithmetic operations of addition, subtraction, multiplication and division can be carried out with rational numbers. We can represent the rationals on a number line as shown in Figure 2.
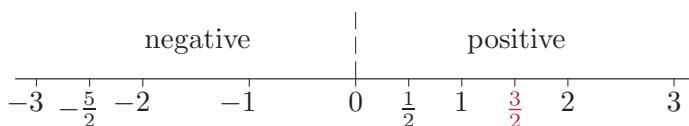
Figure 2   Rational numbers on a number line

For example, the rational $\frac{3}{2}$ is placed at the point which is one half of the way from 0 to 3.

This representation means that rationals have a natural *order* on the number line. For example, $19/22$ lies to the left of $7/8$ because

$$\frac{19}{22} = \frac{76}{88} \quad \text{and} \quad \frac{7}{8} = \frac{77}{88}.$$

If a lies to the left of b on the number line, then we say that

a is *less than* b or b is *greater than* a

and we write

$a < b$ or $b > a.$

For example, we write

$$\frac{19}{22} < \frac{7}{8} \quad \text{or} \quad \frac{7}{8} > \frac{19}{22}.$$

Also, we write $a \le b$ (or $b \ge a$) if either $a < b$ or $a = b$.

Exercise D1

Arrange the following rationals in order:

$$0, \quad 1, \quad -1, \quad \frac{17}{20}, \quad -\frac{17}{20}, \quad \frac{45}{53}, \quad -\frac{45}{53}.$$

Exercise D2

Show that between any two distinct rationals there is another rational.

1.2 Decimal representation of rational numbers

The decimal system enables us to represent all the natural numbers using only the ten integers

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9,$$

which are called *digits*. We now recall the basic facts about the representation of *rational* numbers by decimals.

Definition

A **decimal** is an expression of the form

$$\pm\, a_0.a_1 a_2 a_3 \ldots,$$

where a_0 is a non-negative integer and a_n is a digit for each $n \in \mathbb{N}$.

For example, $13.1212\ldots$ and $-1.111\ldots$ are both decimals. If only a finite number of the digits $a_1, a_2, \ldots$ are non-zero, then the decimal is a **terminating** or **finite decimal**, and we usually omit the tail of zeros. For example, we usually write 0.85 instead of $0.8500\ldots$.

Terminating decimals are used to represent rational numbers in the following way:

$$\pm\, a_0.a_1 a_2 \ldots a_n = \pm \left(a_0 + \frac{a_1}{10^1} + \frac{a_2}{10^2} + \cdots + \frac{a_n}{10^n} \right).$$

For example,

$$0.85 = 0 + \frac{8}{10^1} + \frac{5}{10^2} = \frac{85}{100} = \frac{17}{20}.$$

It can be shown that any fraction for which the factors of the denominator are all powers of 2 and/or 5 (for example, any fraction with denominator $20 = 2^2 \times 5$) can be represented by such a terminating decimal, which can be found by long division; and conversely, that every terminating decimal represents such a fraction.

However, if we apply long division to other rationals, then the process of long division never terminates and we obtain a **non-terminating** or **infinite decimal**. For example, applying long division to $1/3$ gives $0.333\ldots$ and for $19/22$ we obtain $0.863\,63\ldots$.

Exercise D3

Use long division to find the decimal corresponding to $1/7$.

The infinite decimals obtained by applying the long division process to rationals have a certain common property. All of them are **recurring decimals**; that is, they have a repeating block of digits, so they can be written in shorthand form by placing a line over the repeating block, as follows:

$$0.333\ldots \qquad\qquad = 0.\overline{3},$$
$$0.142\,857\,142\,857\ldots = 0.\overline{142\,857},$$
$$0.863\,63\ldots \qquad\quad = 0.8\overline{63}.$$

(Another commonly used notation for recurring decimals is to place a dot over the first and last digits in the repeating block; for example,

$$0.\dot{3} \quad \text{and} \quad 0.\dot{1}42\,85\dot{7}.$$

However, we do not use this notation in this module.)

To see why applying the long division process to a fraction p/q always leads to either a recurring decimal or a terminating decimal, note that there are only q possible remainders at each stage of the division, so one of these remainders must eventually repeat. When this happens, the block of digits obtained after the first occurrence of this remainder will be repeated infinitely often. If the remainder 0 occurs, then the resulting decimal is a terminating decimal; that is, it ends in recurring 0s.

Recurring decimals which arise from the long division of fractions are used to represent the corresponding rational numbers. Conversely, it can be shown that every recurring decimal represents some rational number.

However, the representation of rational numbers by recurring decimals is not quite as straightforward as for terminating decimals. If we try the same approach, we get an equation involving the sum of infinitely many terms, for example,

$$\frac{1}{3} = 0.\overline{3} = \frac{3}{10^1} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots,$$

and it is not immediately clear what such a sum means. This will be made

precise when you have met the idea of the sum of a convergent infinite series later in the book. For the moment, though, when we write the statement $1/3 = 0.\overline{3}$, we mean simply that the decimal $0.\overline{3}$ arises from $1/3$ by the long division process.

The following worked exercise illustrates one way of finding the fraction with a given decimal representation.

Worked Exercise D1

Find the fraction whose decimal representation is $0.8\overline{63}$.

Solution

We begin by finding the fraction whose decimal representation is equal to $0.\overline{63}$.

Let $x = 0.\overline{63}$.

Because the recurring block has length two, we multiply both sides by 10^2.

Multiplying both sides by 10^2, we obtain

$$100x = 63.\overline{63} = 63 + x.$$

Hence

$$99x = 63, \quad \text{so} \quad x = \frac{63}{99} = \frac{7}{11}.$$

Having found x as a fraction, we now write $0.8\overline{63}$ as the sum of two fractions.

Thus

$$0.8\overline{63} = \frac{8}{10} + \frac{x}{10} = \frac{8}{10} + \frac{7}{110} = \frac{95}{110} = \frac{19}{22}.$$

The key idea in the solution above is that multiplication of a decimal by 10^k, where $k \in \mathbb{N}$, moves the decimal point k places to the right.

Exercise D4

Using the method of Worked Exercise D1, find the fractions represented by the following decimals.

(a) $0.\overline{231}$ (b) $2.2\overline{81}$

Decimals which end in recurring 9s sometimes arise as alternative representations for terminating decimals. For example,

$$1 = 0.\overline{9} = 0.999\ldots \quad \text{and} \quad 1.35 = 1.34\overline{9} = 1.349\,99\ldots.$$

You may find this rather disconcerting, but it is important to realise that this representation is a matter of *definition*. We wish to allow the decimal $0.999\ldots$ to represent a number x. This number x must be less than or equal to 1 and greater than each of the numbers

$$0.9, \ 0.99, \ 0.999, \ \ldots.$$

The *only* rational with these properties is 1. When possible, we avoid using the form of a decimal which ends in recurring 9s.

The decimal representation of rational numbers has the advantage that it enables us to decide immediately which of two distinct positive rationals is the greater. We need only examine their decimal representations and notice the first place at which the digits differ. For example, to order $7/8$ and $19/22$, we can write

$$\frac{7}{8} = 0.875 \quad \text{and} \quad \frac{19}{22} = 0.863\,63\ldots.$$

Then

$$\begin{array}{cc} \downarrow & \downarrow \\ 0.863\,63\ldots < 0.875, & \text{so} \quad 19/22 < 7/8. \end{array}$$

Note that, when comparing decimals in this way, the decimal on the left should not end in recurring 9s unless the decimal on the right is also non-terminating.

Exercise D5

Find the first two digits after the decimal point in the decimal representations of $17/20$ and $45/53$, and hence determine which of these two rationals is the greater.

1.3 Irrational numbers

You saw in Unit A2 *Number systems* that there is no rational number which satisfies the equation $x^2 = 2$; that is, $\sqrt{2}$ is not rational.

There are many other mathematical quantities that cannot be described exactly by rational numbers. For example:

- if m and n are natural numbers, and the equation $x^m = n$ has no integer solution, then the positive solution of this equation, written as $\sqrt[m]{n}$, is not rational

- the number π, which denotes the ratio of the circumference of a circle to its diameter

- the number e, the base of natural logarithms.

A number which is not rational is called **irrational**. It is natural to ask whether irrational numbers, such as $\sqrt{2}$ and π, can be represented as decimals. Using your calculator, you can check that

$$(1.414\,213\,56)^2$$

is very close to 2, so $1.414\,213\,56$ is a very good approximate value for $\sqrt{2}$. But is there a decimal that represents $\sqrt{2}$ exactly?

In fact, it is possible to represent all irrational numbers by **non-recurring decimals**, that is, infinite decimals that do not end in a recurring block of digits. For example, there are non-recurring decimals representing $\sqrt{2}$ and π, the first few digits of which are

$$\sqrt{2} = 1.414\,213\,56\ldots \quad \text{and} \quad \pi = 3.141\,592\,65\ldots.$$

Conversely, it is also natural to ask whether arbitrary non-recurring decimals, such as

$$0.101\,001\,000\,100\,001\ldots \quad \text{and} \quad 0.123\,456\,789\,101\,112\ldots,$$

always represent irrational numbers. We take it as a basic assumption about the number system that they do.

Thus the set of irrational numbers consists of all the non-recurring decimals.

1.4   Real numbers and their properties

We can now define what we mean by the set of real numbers.

> **Definition**
>
> The set of **real numbers**, denoted by $\mathbb{R}$, is the union of the set of rational numbers and the set of irrational numbers. In other words, it is the set of all terminating, recurring and non-recurring decimals.

As with rational numbers, we can determine which of two real numbers is greater by comparing their decimals and noticing the first pair of corresponding digits which differ. For example,

$$\begin{array}{cc} \downarrow & \downarrow \\ 0.101\,001\,000\,100\,001\ldots & < \; 0.123\,456\,789\,101\,112\ldots. \end{array}$$

We now use this idea to associate with each irrational number a point on the number line. For example, as illustrated in Figure 3, the irrational number whose decimal representation begins

$$x = 0.123\,456\,789\,101\,112\ldots$$

satisfies each of the inequalities

$$0.1 < x < 0.2$$
$$0.12 < x < 0.13$$
$$0.123 < x < 0.124$$
$$\vdots$$

Figure 3   The point on the number line corresponding to $x = 0.123\,45\ldots$

We assume that there is a point on the number line corresponding to x which lies to the right of each of the rational numbers 0.1, 0.12, 0.123, ... and to the left of each of the rational numbers 0.2, 0.13, 0.124, ...

As usual, negative real numbers correspond to points lying to the left of 0. The number line, complete with both rational and irrational points, is called the **real line**; see Figure 4.
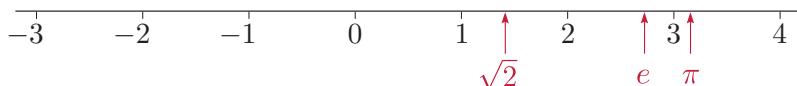
Figure 4   The real line

Our definition of the real numbers is therefore consistent with the picture of the real numbers as a number line, which you met in Unit A2. Saying the same thing more formally, there is a one-to-one correspondence between the points on the real line and the set $\mathbb{R}$ of real numbers, as defined above.

In the box below, we list several *order properties* of $\mathbb{R}$. You are probably already familiar with these, though you may not have met their names before.

Order properties of $\mathbb{R}$

Trichotomy Property   If $a, b \in \mathbb{R}$, then *exactly one* of the
following holds:

$$a < b \quad \text{or} \quad a = b \quad \text{or} \quad a > b.$$

Transitive Property   If $a, b, c \in \mathbb{R}$, then

$$a < b \text{ and } b < c \implies a < c.$$

Archimedean Property   If $a \in \mathbb{R}$, then there is a positive integer n
such that

$$n > a.$$

Density Property   If $a, b \in \mathbb{R}$ and $a < b$, then there is a rational
number x and an irrational number y such that

$$a < x < b \quad \text{and} \quad a < y < b.$$

These properties are used frequently in analysis, though they are not often
referred to explicitly by name. In this unit, however, we often use the
names to point out when the properties are being used, to aid your
understanding.

Each of the order properties in the above box can be proved from our
definition of the real numbers, but we do not give the proofs here. The
first three are almost self-evident, but the Density Property is not so
obvious. One consequence of the Density Property is that between any two
distinct real numbers there are infinitely many rational numbers and
infinitely many irrational numbers. The next worked exercise gives an
example of how to find a rational number and an irrational number
between two given real numbers. The method is indicative of how the
Density Property could be proved for general real numbers a and b.

Worked Exercise D2

Let $a = 0.12\overline{3}$ and $b = 0.123\,45\ldots$. Find a rational number x and an irrational number y such that

$$a < x < b \quad \text{and} \quad a < y < b.$$

Solution

💭 There are many different methods you could use. One method is to begin by noting that the two decimals

$$a = 0.123\,33\ldots \quad \text{and} \quad b = 0.123\,45\ldots$$

differ first at the fourth digit after the decimal point. If we truncate b after this digit, then we obtain a suitable rational number. 💭

For example, the rational number

$$x = 0.1234$$

satisfies $a < x < b$.

💭 To find an irrational number y between a and b, we attach to x a (sufficiently small) non-recurring tail, such as $010\,010\,001\ldots$. The resulting number is irrational since its decimal representation is non-recurring. 💭

The irrational number

$$y = 0.1234\,|\,010010001\ldots$$
$$|\,\text{non-recurring tail}$$

satisfies $a < y < b$.

Exercise D6

Let $a = 0.\overline{3}$ and $b = 0.3401$. Find a rational number x and an irrational number y such that $a < x < b$ and $a < y < b$.

1.5   Arithmetic with real numbers

We now turn to the question of how to carry out arithmetic using real numbers, bearing in mind our definition of the real numbers in the last subsection.

You saw in Unit A2 that the set $\mathbb{R}$ of real numbers, in common with the set $\mathbb{Q}$ of rational numbers, forms a *field*. This means that the properties in the box below hold for arithmetic in $\mathbb{R}$.

Arithmetic in $\mathbb{R}$

Properties for addition

A1 Closure   For all $a, b \in \mathbb{R}$,

$$a + b \in \mathbb{R}.$$

A2 Associativity   For all $a, b, c \in \mathbb{R}$,

$$a + (b + c) = (a + b) + c.$$

A3 Additive identity   For all $a \in \mathbb{R}$,

$$a + 0 = a = 0 + a.$$

A4 Additive inverses   For each $a \in \mathbb{R}$, there is a number $-a \in \mathbb{R}$ such that

$$a + (-a) = 0 = (-a) + a.$$

A5 Commutativity   For all $a, b \in \mathbb{R}$,

$$a + b = b + a.$$

Properties for multiplication

M1 Closure   For all $a, b \in \mathbb{R}$,

$$a \times b \in \mathbb{R}.$$

M2 Associativity   For all $a, b, c \in \mathbb{R}$,

$$a \times (b \times c) = (a \times b) \times c.$$

M3 Multiplicative identity   For all $a \in \mathbb{R}$,

$$a \times 1 = a = 1 \times a.$$

M4 Multiplicative inverses   For each $a \in \mathbb{R}^*$, there is a number $a^{-1} \in \mathbb{R}$ such that

$$a \times a^{-1} = 1 = a^{-1} \times a.$$

M5 Commutativity   For all $a, b \in \mathbb{R}$,

$$a \times b = b \times a.$$

Property combining addition and multiplication

D1 Distributivity   For all $a, b, c \in \mathbb{R}$,

$$a \times (b + c) = a \times b + a \times c.$$

Put more succinctly, the properties in the box mean that:

- $\mathbb{R}$ is an abelian group under the operation of addition $+$
- $\mathbb{R}^* = \mathbb{R} - \{0\}$ is an abelian group under the operation of multiplication $\times$;

and these two group structures are linked by the distributive property.

In Unit A2, these properties were introduced simply as a way of formalising the elementary rules of arithmetic that were already familiar to you. However, now that we have defined the real numbers as the set of all terminating, recurring and non-recurring decimals, we need to show that these properties follow from the definition.

For terminating and recurring decimals (that is, the rational numbers), this is straightforward: we can do arithmetic by first converting the decimals to the corresponding fractions. However, it is not obvious how to do arithmetic with non-recurring decimals (that is, irrationals). For example, assuming that we can represent $\sqrt{2}$ and π by the non-recurring decimals that begin

$$\sqrt{2} = 1.414\,213\,56\ldots \quad \text{and} \quad \pi = 3.141\,592\,65\ldots,$$

can we also represent the sum $\sqrt{2} + \pi$ and the product $\sqrt{2} \times \pi$ as decimals? In other words, what is meant by the operations of addition and multiplication when non-recurring decimals are involved, and do these operations satisfy the properties stated in the above box?

It turns out that setting up the appropriate definitions and proving the necessary properties is a lengthy process, and we will not go into this here. The important point is that, using our definition of the real numbers, addition and multiplication *can* be formally defined, and it can be proved that all the above properties hold in $\mathbb{R}$. From now on, we *assume* that this process has been carried out. Furthermore, we assume that the set $\mathbb{R}$ contains the nth roots and rational powers of positive real numbers, with their usual properties. We describe one way to justify these assumptions in Section 5.

We conclude this section by noting that analysis texts take various approaches to defining the real numbers. For example, it is common to assume that there exists a set $\mathbb{R}$ which is a field containing $\mathbb{Q}$ and having certain extra properties, and then to deduce all results from these assumptions. In this 'axiomatic approach' the definition of the real numbers themselves may not be given (though they can be defined by a somewhat abstract procedure involving partitions of $\mathbb{Q}$ called 'Dedekind cuts'), but it is then *proved* that each real number must have a decimal representation. In this module, we adopt a more concrete approach in which the real numbers are *defined* to be decimals.

2   Inequalities

Much of analysis is concerned with inequalities of various kinds; the aim of this section and the next is to provide practice in their manipulation. In this section you will meet the rules for manipulating inequalities and see how to solve inequalities by using these rules.

2.1   Rearranging inequalities

The fundamental rule, on which much manipulation of inequalities is based, is that the statement $a < b$ means exactly the same as the statement $b - a > 0$. We express this as follows.

Rule 1

$a < b \iff b - a > 0.$

Recall that the symbol $\iff$ means 'if and only if' or 'is equivalent to'. Thus, put another way, this rule says that the inequalities $a < b$ and $b - a > 0$ are equivalent.

There are several other standard rules for rearranging an inequality into an equivalent form. Each of these can be deduced from Rule 1, although the proofs are not given here. For example, we obtain an equivalent inequality by adding the same expression to both sides.

Rule 2

$a < b \iff a + c < b + c.$

Another way to rearrange an inequality is to multiply both sides by an expression that is strictly greater than zero. It is also possible to multiply both sides by an expression that is strictly less than zero, but in this case the inequality must be reversed. For example, $2 < 3 \iff -2 > -3$.

Rule 3

If $c > 0$, then

$\qquad a < b \iff ac < bc;$

if $c < 0$, then

$\qquad a < b \iff ac > bc.$

Sometimes the most effective way to rearrange an inequality is to take reciprocals. You can do this if both sides of the inequality are positive, and it is important to remember that the direction of the inequality has to be reversed. For example, $2 < 3 \iff \frac{1}{2} > \frac{1}{3}$.

Rule 4

If $a, b > 0$, then

$$a < b \iff \frac{1}{a} > \frac{1}{b}.$$

Some inequalities can be simplified only by taking powers. In order to do this, both sides must be non-negative and the power must be positive.

Rule 5

If $a, b \geq 0$ and $p > 0$, then

$$a < b \iff a^p < b^p.$$

For positive integers p, Rule 5 follows from the identity

$$b^p - a^p = (b - a)(b^{p-1} + b^{p-2}a + \cdots + ba^{p-2} + a^{p-1});$$

since the value of the right-hand bracket is positive, we deduce that

$$b - a > 0 \iff b^p - a^p > 0.$$

For other positive real numbers p the proof of Rule 5 is harder, but the rule remains true.

There are corresponding versions of Rules 1–5 in which the *strict* inequality $a < b$ is replaced by the *weak* inequality $a \leq b$. For example, if $c > 0$, then

$$a \leq b \iff ac \leq bc.$$

The box below summarises the rules you have met so far for manipulating inequalities.

Rules for rearranging inequalities

Let a, b, c and p be real numbers.

Rule 1 $a < b \iff b - a > 0.$

Rule 2 $a < b \iff a + c < b + c.$

Rule 3   If $c > 0$, then $a < b \iff ac < bc$;
            if $c < 0$, then $a < b \iff ac > bc.$

Rule 4   If $a, b > 0$, then

$$a < b \iff \frac{1}{a} > \frac{1}{b}.$$

Rule 5   If $a, b \geq 0$ and $p > 0$, then

$$a < b \iff a^p < b^p.$$

All the above rules also hold if strict inequalities are replaced by weak inequalities.

In manipulating inequalities, we also make frequent use of the usual rules for the sign of a product:

$\times$	$+$	$-$
$+$	$+$	$-$
$-$	$-$	$+$

In particular, the square of any real number is non-negative.

The next exercise gives you a chance to practise using the rules for rearranging inequalities.

Exercise D7

In each of the following cases, apply the rules to the inequality $x > 2$ to obtain an equivalent inequality that contains the given expression, noting carefully which rules you are using.

(a) $x + 3$ (b) $2 - x$ (c) $5x + 2$ (d) $1/(5x + 2)$

2.2 Solving inequalities

Solving an inequality that involves an unknown real number x means determining the values of x for which the inequality is true; that is, finding the **solution set** of the inequality, usually as a union of intervals. We can often do this by rewriting the inequality in an equivalent but simpler form, using Rules 1–5.

Worked Exercise D3

Solve the inequality
$$\frac{x + 2}{x + 4} > \frac{x - 3}{2x - 1}.$$

Solution

Observe that we *cannot* solve this inequality by cross-multiplying (that is, by multiplying both sides by the product of both denominators), because the denominators have different signs or are zero for some values of x, so we are unable to apply Rule 3. Instead, we rearrange the inequality using Rule 1, to give an equivalent inequality with just 0 on one side.

Rearranging the inequality gives
$$\frac{x + 2}{x + 4} > \frac{x - 3}{2x - 1} \iff \frac{x + 2}{x + 4} - \frac{x - 3}{2x - 1} > 0$$
$$\iff \frac{x^2 + 2x + 10}{(x + 4)(2x - 1)} > 0.$$

By completing the square, we obtain

$$x^2 + 2x + 10 = (x+1)^2 + 9,$$

so the numerator is always positive.

We can now find the solution set using a table of signs:

x	$(-\infty, -4)$	-4	$\left(-4, \frac{1}{2}\right)$	$\frac{1}{2}$	$\left(\frac{1}{2}, \infty\right)$
$x^2 + 2x + 10$	$+$	$+$	$+$	$+$	$+$
$x + 4$	$-$	0	$+$	$+$	$+$
$2x - 1$	$-$	$-$	$-$	0	$+$
$\dfrac{x^2 + 2x + 10}{(x+4)(2x-1)}$	$+$	$*$	$-$	$*$	$+$

So the solution set is

$$\left\{ x : \frac{x+2}{x+4} > \frac{x-3}{2x-1} \right\} = (-\infty, -4) \cup \left(\tfrac{1}{2}, \infty\right).$$

We now consider an inequality that *could* be solved by using cross-multiplication, though in this case it is easier to take reciprocals using Rule 4.

Worked Exercise D4

Solve the inequality

$$\frac{1}{2x^2 + 2} < \frac{1}{4}.$$

Solution

💭 Since both sides of the inequality are always positive, we can apply Rule 4. 💭

Since $2x^2 + 2 > 0$, we can rearrange the inequality into an equivalent form as follows:

$$
\begin{aligned}
\frac{1}{2x^2 + 2} < \frac{1}{4} \;&\Longleftrightarrow\; 2x^2 + 2 > 4 \quad \text{(by Rule 4)} \\
&\Longleftrightarrow\; x^2 + 1 > 2 \quad \text{(by Rule 3)} \\
&\Longleftrightarrow\; x^2 - 1 > 0 \quad \text{(by Rule 1)} \\
&\Longleftrightarrow\; (x-1)(x+1) > 0.
\end{aligned}
$$

💭 At this point, it might be helpful to write down a table of signs or, alternatively, quickly sketch the parabola that is the graph of $y = (x-1)(x+1)$. 💭
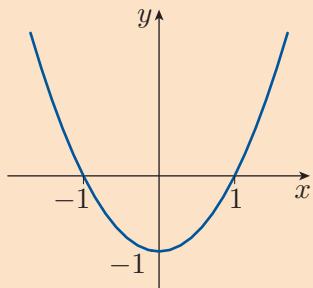
So the solution set is

$$\left\{ x : \frac{1}{2x^2 + 2} < \frac{1}{4} \right\} = (-\infty, -1) \cup (1, \infty).$$

Exercise D8

Solve the following inequalities.

(a) $\dfrac{4x - x^2 - 7}{x^2 - 1} \geq 3$ (b) $2x^2 \geq (x+1)^2$

We now consider an inequality which involves rational powers. Here we need to be careful, when applying Rule 5, to ensure that both sides of the inequality are non-negative.

Worked Exercise D5

Solve the inequality

$$\sqrt{2x + 3} > x.$$

Solution

We can get rid of the awkward square root sign in the inequality by squaring both sides – that is, by applying Rule 5 with $p = 2$. However, we can do this only if both sides are non-negative. Remember that $\sqrt{}$ always means the non-negative square root.

The expression $\sqrt{2x + 3}$ is defined only when $2x + 3 \geq 0$, that is, for $x \geq -3/2$. Hence we need only consider those x in $[-3/2, \infty)$.

For $x \geq 0$, we have

$$\begin{aligned}
\sqrt{2x + 3} > x &\iff 2x + 3 > x^2 \quad \text{(by Rule 5, with } p = 2) \\
&\iff x^2 - 2x - 3 < 0 \quad \text{(by Rule 1)} \\
&\iff (x - 3)(x + 1) < 0.
\end{aligned}$$

So the part of the solution set in $[0, \infty)$ is $[0, 3)$.

On the other hand, if $-3/2 \leq x < 0$, then

$$\sqrt{2x + 3} \geq 0 > x,$$

so the other part of the solution set is $[-3/2, 0)$.

Hence the complete solution set is

$$\left\{ x : \sqrt{2x + 3} > x \right\} = [-3/2, 0\,) \cup [0, 3) = [-3/2, 3).$$

Exercise D9

Solve the inequality

$$\sqrt{2x^2 - 2} > x.$$

2.3   Inequalities involving modulus signs

Now we consider inequalities involving the *modulus* of a real number. Recall that if $a \in \mathbb{R}$, then its **modulus**, or **absolute value**, $|a|$ is defined by

$$|a| = \begin{cases} a, & \text{if } a \geq 0, \\ -a, & \text{if } a < 0. \end{cases}$$

Figure 5   The graph of $y = |x|$
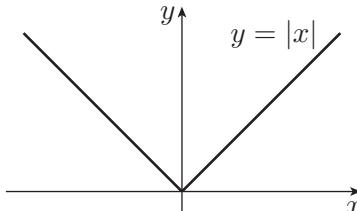
The graph of $y = |x|$ is illustrated in Figure 5.

It is useful to think of $|a|$ as the distance along the real line from 0 to a. For example, $|3| = |-3| = 3$ as shown in Figure 6.
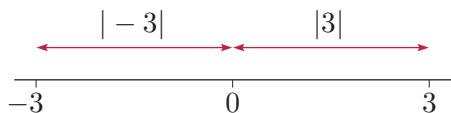
Figure 6   The modulus of 3 and -3

In the same way, $|a - b|$ is the distance from 0 to $a - b$, which is the same as the distance from a to b, as illustrated in Figure 7. For example, the distance from -2 to 3 is $|(-2) - 3| = |-5| = 5$.
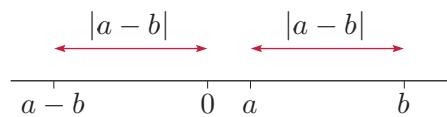
Figure 7   The modulus of $a - b$

Note also that $|a + b| = |a - (-b)|$ is the distance from a to $-b$, as illustrated in Figure 8.
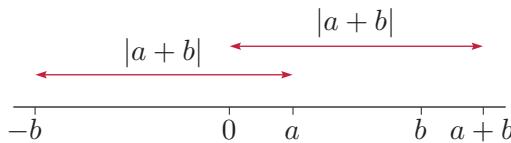
Figure 8   The modulus of $a + b$

We list below some basic properties of the modulus, which follow immediately from the definition.

Properties of the modulus

If $a, b \in \mathbb{R}$, then

1. $|a| \geq 0$, with equality if and only if $a = 0$
2. $-|a| \leq a \leq |a|$
3. $|a|^2 = a^2$
4. $|a - b| = |b - a|$
5. $|ab| = |a||b|$.

We now give a rule for rearranging inequalities that involve the modulus of an expression. The rule follows from property 2 in the above list, and as usual there is a corresponding version with weak inequalities.

Rule 6

$|a| < b \iff -b < a < b.$

You can use this rule alongside the other rules for rearranging inequalities that you met in Subsection 2.1. For reference, here is a summary of all the rules.

Rules for rearranging inequalities

Let a, b, c and p be real numbers.

Rule 1 $a < b \iff b - a > 0.$

Rule 2 $a < b \iff a + c < b + c.$

Rule 3   If $c > 0$, then $a < b \iff ac < bc$;
         if $c < 0$, then $a < b \iff ac > bc.$

Rule 4   If $a, b > 0$, then

$$a < b \iff \frac{1}{a} > \frac{1}{b}.$$

Rule 5   If $a, b \geq 0$ and $p > 0$, then
$$a < b \iff a^p < b^p.$$

Rule 6 $|a| < b \iff -b < a < b.$

All the above rules also hold if strict inequalities are replaced by weak inequalities.

Worked Exercise D6

Solve the inequality $|x - 2| < 1$.

Solution

We rearrange the inequality into an equivalent form by using Rule 6 with $a = x - 2$ and $b = 1$.

We have
$$|x - 2| < 1 \iff -1 < x - 2 < 1 \quad \text{(by Rule 6)}$$
$$\iff 1 < x < 3.$$

So the solution set is
$$\{x : |x - 2| < 1\} = (1, 3).$$

We can also rearrange inequalities involving modulus signs by using Rule 5 with $p = 2$, since the modulus of a number is non-negative.

Worked Exercise D7

Solve the inequality $|x - 2| \leq |x + 1|$.

Solution

We rearrange the inequality into an equivalent form by using Rule 5 with $a = |x - 2|$, $b = |x + 1|$ and $p = 2$.

We have
$$|x - 2| \leq |x + 1| \iff (x - 2)^2 \leq (x + 1)^2 \text{ (by Rule 5, with } p = 2)$$
$$\iff x^2 - 4x + 4 \leq x^2 + 2x + 1$$
$$\iff 3 \leq 6x$$
$$\iff \tfrac{1}{2} \leq x.$$

So the solution set is
$$\{x : |x - 2| \leq |x - 1|\} = [\tfrac{1}{2}, \infty).$$

Thinking about what an inequality means geometrically can often give you an idea of its solution set.

For example, the inequalities in Worked Exercises D6 and D7 can be interpreted geometrically as follows.

The inequality

$$|x - 2| < 1$$

holds when the distance from x to 2 is strictly less than 1. So it holds when x lies in the open interval $(1, 3)$, which has midpoint 2. This is what we found in Worked Exercise D6 by using Rule 6.

Similarly, the inequality

$$|x - 2| \leq |x + 1|$$

holds when the distance from x to 2 is less than or equal to the distance from x to -1, since $|x + 1| = |x - (-1)|$. As the point halfway from -1 to 2 is $\frac{1}{2}$, the inequality holds when x lies in the interval $\left[\frac{1}{2}, \infty\right)$, as illustrated in Figure 9. This agrees with what we found algebraically in Worked Exercise D7.

$$\{x : |x - 2| \leq |x + 1|\}$$
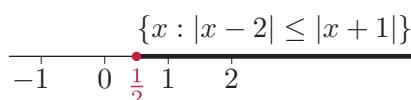
Figure 9   The solution set of $|x - 2| \leq |x + 1|$

Exercise D10

Solve the following inequalities.

(a) $|2x^2 - 13| < 5$ (b) $|x - 1| \leq 2|x + 1|$

3   Proving inequalities

In this section you will see how to *prove* inequalities of various types. Several of the inequalities we prove here will be used in later analysis units of this module.

In proving inequalities, we will make use of the rules for rearranging inequalities that you met in Section 2, together with some further rules for deducing 'new inequalities from old', which we outline here.

You met the first such rule in Section 1, where it was called the Transitive Property of $\mathbb{R}$.

Transitive Rule for inequalities

$a < b$ and $b < c$ $\implies$ $a < c$.

We use the Transitive Rule when we want to prove that $a < c$ and we know that $a < b$ and $b < c$.

The following rules are also useful.

> **Combination Rules for inequalities**
>
> If $a < b$ and $c < d$, then
>
> **Sum Rule** $a + c < b + d$
>
> **Product Rule** $ac < bd$, provided $a, c \geq 0$.

For example, since $2 < 3$ and $4 < 5$, it follows that

$$2 + 4 < 3 + 5,$$
$$2 \times 4 < 3 \times 5.$$

There are versions of the Transitive Rule and Combination Rules involving weak inequalities. For example, a weak version of the Transitive Rule is

$$a \leq b \text{ and } b \leq c \implies a \leq c.$$

Notice that the Transitive Rule and the Combination Rules have a different nature from Rules 1–6 given in Section 2. Rules 1–6 tell us how to rearrange inequalities into *equivalent* inequalities, whereas the Transitive Rule and the Combination Rules enable us to start with two inequalities and deduce a new inequality which is *not* equivalent to either of the old ones, but does follow from them. For example, if we know that $x < y$ and $y < 5$, then we can use the Transitive Rule to deduce that $x < 5$; this new inequality is not equivalent to either of the inequalities that we started with, but it does follow from them.

3.1   Triangle Inequality

Now we meet an inequality that can be used to deduce 'new inequalities from old', but is also of great importance in its own right.

This inequality involves the modulus of three real numbers a, b and $a + b$, and is called the *Triangle Inequality* because it is related to the fact that the length of one side of a triangle is less than the sum of the lengths of the other two sides. As you will see, the Triangle Inequality has many applications in analysis.

> **Triangle Inequality**
>
> If $a, b \in \mathbb{R}$, then
>
> 1. $|a + b| \leq |a| + |b|$ (usual form)
> 2. $|a - b| \geq \big||a| - |b|\big|$ ('backwards' form).

Proof

1. We rearrange the inequality into an equivalent form:
$$|a + b| \leq |a| + |b| \iff (a+b)^2 \leq (|a| + |b|)^2 \quad \text{(by Rule 5, with } p = 2)$$
$$\iff a^2 + 2ab + b^2 \leq a^2 + 2|a|\,|b| + b^2$$
$$\iff 2ab \leq |2ab|.$$

   This final inequality is certainly true for all $a, b \in \mathbb{R}$, so the first inequality must also be true for all $a, b \in \mathbb{R}$.

2. We prove the backwards form using the same method:
$$|a - b| \geq ||a| - |b|| \iff (a-b)^2 \geq (|a| - |b|)^2 \text{ (by Rule 5, with } p = 2)$$
$$\iff a^2 - 2ab + b^2 \geq a^2 - 2|a|\,|b| + b^2$$
$$\iff -2ab \geq -|2ab|$$
$$\iff 2ab \leq |2ab|,$$

   which, as before, is true for all $a, b \in \mathbb{R}$. ∎

Remarks

1. Although we have used implications in both directions here, the proof requires only the implications going from right to left. For example, in the proof of the usual form of the Triangle Inequality, the important implication is

   $$|a + b| \leq |a| + |b| \impliedby 2ab \leq |2ab|.$$

2. The usual form of the Triangle Inequality can also be proved by using Rule 6, which gives
   $$|a + b| \leq |a| + |b| \iff -(|a| + |b|) \leq a + b \leq |a| + |b|. \tag{1}$$
   Now since we know that

   $$-|a| \leq a \leq |a| \quad \text{and} \quad -|b| \leq b \leq |b|,$$

   it follows from the Sum Rule that

   $$-(|a| + |b|) \leq a + b \leq |a| + |b|,$$

   which is the statement on the right in (1). Hence the equivalent statement on the left in (1) also holds, and this is the Triangle Inequality:

   $$|a + b| \leq |a| + |b|.$$

3. There is a version of the Triangle Inequality for n real numbers, where $n \geq 2$:

   $$|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|.$$

   This can be proved using the Principle of Mathematical Induction, which you met in Book A and will use again in Subsection 3.5 of this unit.

The worked exercise below illustrates some typical applications of the Triangle Inequality. Remember that, in each part, we are deducing one inequality from another, *not* showing that two inequalities are equivalent.

Worked Exercise D8

Use the Triangle Inequality to prove the following statements.

(a) $|a| \leq 1 \implies |3 + a^3| \leq 4$ (b) $|b| < 1 \implies |3 - b| > 2$

Solution

(a) Suppose that $|a| \leq 1$. The Triangle Inequality gives

$$|3 + a^3| \leq |3| + |a^3|$$
$$= 3 + |a|^3.$$

Now $|a| \leq 1$ and therefore

$$3 + |a|^3 \leq 3 + 1 = 4.$$

We deduce, using the Transitive Rule, that

$$|a| \leq 1 \implies |3 + a^3| \leq 4.$$

(b) Suppose that $|b| < 1$. The backwards form of the Triangle Inequality gives

$$|3 - b| \geq \big||3| - |b|\big|$$
$$= |3 - |b||$$
$$\geq 3 - |b|.$$

Now $|b| < 1$, so $-|b| > -1$, and hence

$$3 - |b| > 3 - 1 = 2.$$

We deduce, using the previous chain of inequalities, that

$$|b| < 1 \implies |3 - b| > 2.$$

Remarks

1. The statements proved in Worked Exercise D8 can also be written in the following form:

$$|3 + a^3| \leq 4, \quad \text{for } |a| \leq 1$$

and

$$|3 - b| > 2, \quad \text{for } |b| < 1.$$

2. Notice that the reverse implications

$$|3 + a^3| \leq 4 \implies |a| \leq 1 \quad \text{and} \quad |3 - b| > 2 \implies |b| < 1$$

are *false*. For example, try putting $a = -\frac{3}{2}$ and $b = -2$.

Use the Triangle Inequality to prove the following statements.

(a) $|a| \leq \frac{1}{2} \implies |a+1| \leq \frac{3}{2}$ (b) $|b| < \frac{1}{2} \implies |b^3 - 1| > \frac{7}{8}$

3.2   Proving inequalities by rearrangement

We now prove some further inequalities using the method you saw in the proof of the Triangle Inequality. As in that proof, we start from the inequality we wish to prove, and use the various rearrangement rules to obtain a chain of equivalent inequalities until we reach an inequality that we know must be true.

In the worked exercises in this subsection, we apply this technique to prove two inequalities with geometric interpretations. The first states that

$$ab \leq \left(\frac{a+b}{2} \right)^2, \quad \text{for } a, b \in \mathbb{R}.$$

This inequality has the geometric interpretation illustrated in Figure 10: the area of a rectangle with sides of length a and b is less than or equal to the area of a square with the same perimeter, that is, with sides of length $(a+b)/2$.
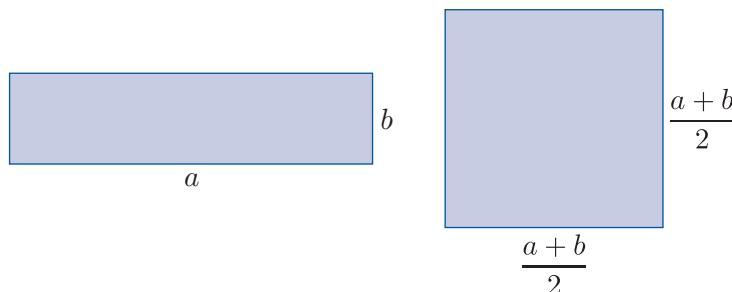

Figure 10   Comparison between the area of a rectangle and the area of a square with the same perimeter

In the worked exercises we include some comments about which of the rearrangement rules we are using, but you need not do this when you write out solutions.

Worked Exercise D9

Prove that

$$ab \leq \left(\frac{a+b}{2}\right)^2, \quad \text{for } a, b \in \mathbb{R}.$$

Solution

🗨 We begin by multiplying out the bracket to see if this helps to simplify the inequality. We then apply the rearrangement rules. 🗨

Rearranging the inequality, we obtain:

$$ab \leq \left(\frac{a+b}{2}\right)^2 \iff ab \leq \frac{a^2 + 2ab + b^2}{4}$$

$$\iff 4ab \leq a^2 + 2ab + b^2 \qquad \text{(by Rule 3)}$$

$$\iff 0 \leq a^2 - 2ab + b^2 \qquad \text{(by Rule 1)}$$

$$\iff 0 \leq (a-b)^2.$$

This final inequality is true, since the square of every real number is non-negative. It follows that the original inequality $ab \leq \left(\frac{a+b}{2}\right)^2$ is also true, for $a, b \in \mathbb{R}$.

If you examine the chain of equivalent statements in the solution to Worked Exercise D9, then you will see that we can replace the symbol $\leq$ with $=$ throughout, without upsetting the equivalence of the statements. It follows that

$$ab = \left(\frac{a+b}{2}\right)^2 \text{ if and only if } a = b.$$

This tells us that the maximum area is obtained when the rectangle is a square.

In the next worked exercise you will see a proof of the following inequality:

$$\sqrt{a^2 + b^2} \leq a + b, \quad \text{for } a, b \geq 0.$$

This inequality has the geometric interpretation illustrated in Figure 11: the length of the hypotenuse of a right-angled triangle whose other sides are of lengths a and b is less than or equal to the sum of the lengths of those two sides.
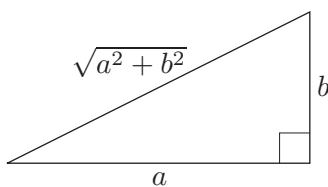

Figure 11   The lengths of the sides of a right-angled triangle

Worked Exercise D10

Prove that
$$\sqrt{a^2 + b^2} \le a + b, \quad \text{for } a, b \ge 0.$$

Solution

🔍 The first step is to remove the awkward square root by applying Rule 5, which we can do because $a, b \ge 0$. 💬

Rearranging the inequality, we obtain:
$$
\begin{aligned}
\sqrt{a^2 + b^2} \le a + b \;&\Longleftrightarrow\; a^2 + b^2 \le (a+b)^2 \text{ (by Rule 5, with } p = 2) \\
&\Longleftrightarrow\; a^2 + b^2 \le a^2 + 2ab + b^2 \\
&\Longleftrightarrow\; 0 \le 2ab. \quad \text{(by Rule 1)}
\end{aligned}
$$

This final inequality is certainly true, since $a, b \ge 0$. It follows that the original inequality $\sqrt{a^2 + b^2} \le a + b$ is also true, for $a, b \ge 0$.

We can reformulate the inequality in Worked Exercise D10 in the following alternative way, which is sometimes useful in applications. If we write c in place of a^2 and d in place of b^2, then the inequality becomes
$$\sqrt{c + d} \le \sqrt{c} + \sqrt{d}, \quad \text{for } c, d \ge 0. \tag{2}$$
As an application, we use this new form of the inequality to prove that
$$\left| \sqrt{a} - \sqrt{b} \right| \le \sqrt{|a - b|}, \quad \text{for } a, b \ge 0. \tag{3}$$

First note that interchanging the roles of a and b leaves inequality (3) unaltered, so it is sufficient to prove the inequality under the assumption that $a \ge b$. It follows from this assumption that $|a - b| = a - b$, and also that $\sqrt{a} \ge \sqrt{b}$, so that $\left| \sqrt{a} - \sqrt{b} \right| = \sqrt{a} - \sqrt{b}$. Hence
$$
\begin{aligned}
\left| \sqrt{a} - \sqrt{b} \right| \le \sqrt{|a - b|} \;&\Longleftrightarrow\; \sqrt{a} - \sqrt{b} \le \sqrt{a - b} \\
&\Longleftrightarrow\; \sqrt{a} \le \sqrt{a - b} + \sqrt{b} \quad \text{(by Rule 2).}
\end{aligned}
$$

This final inequality is certainly true, since it can be obtained from inequality (2) simply by substituting $a - b$ in place of c, and b in place of d. It follows that inequality (3) is indeed true.

Exercise D12

Suppose that $a > 0$ and $a^2 > 2$. Prove that
$$\frac{1}{2}\left(a + \frac{2}{a}\right) < a.$$

3.3   Inequalities involving integers

In analysis we often need to prove inequalities involving an integer n. It is a common convention in mathematics that the symbol n is used to denote an integer, frequently a natural number. So, for example, the expression $n \geq 3$ means $n = 3, 4, 5, \ldots$.

It is often possible to deal with inequalities involving integers by using the rules of rearrangement, just as we did in Subsection 3.2. The next worked exercise gives an example, and then there are two exercises for you to try.

Worked Exercise D11

Prove that

$$2n^2 \geq (n+1)^2, \quad \text{for } n \geq 3.$$

Solution

Rearranging this inequality into an equivalent form, we obtain
$$
\begin{aligned}
2n^2 \geq (n+1)^2 \ &\Longleftrightarrow \ 2n^2 - (n+1)^2 \geq 0 \quad \text{(by Rule 1)} \\
&\Longleftrightarrow \ n^2 - 2n - 1 \geq 0 \\
&\Longleftrightarrow \ (n-1)^2 - 2 \geq 0 \quad \text{(completing the square)} \\
&\Longleftrightarrow \ (n-1)^2 \geq 2. \quad \text{(by Rule 1)}
\end{aligned}
$$

This final inequality is true for $n \geq 3$, so the original inequality $2n^2 \geq (n+1)^2$ is also true for $n \geq 3$.

An alternative solution to Worked Exercise D11 is the following:
$$
\begin{aligned}
2n^2 \geq (n+1)^2 \ &\Longleftrightarrow \ 2 \geq \left(\frac{n+1}{n}\right)^2 \quad \text{(by Rule 3)} \\
&\Longleftrightarrow \ \sqrt{2} \geq 1 + \frac{1}{n} \quad \text{(by Rule 5, with } p = \tfrac{1}{2}\text{).}
\end{aligned}
$$

This final inequality certainly holds for $n \geq 3$, and so the original inequality holds too.

Exercise D13

Prove that

$$\frac{3n}{n^2 + 2} < 1, \quad \text{for } n > 2.$$

Exercise D14

Prove that

$$2n^3 \geq (n+1)^3, \quad \text{for } n \geq 4.$$

3.4 The Binomial Theorem

We often use a result known as the Binomial Theorem to prove inequalities involving integers. The Binomial Theorem gives us a general formula for the expansion of $(a + b)^n$, where $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$. You will already be familiar with the special case when $n = 2$:

$$(a + b)^2 = a^2 + 2ab + b^2.$$

The statement of the Binomial Theorem uses the notation in the box below. Remember that the product $n \times (n - 1) \times \cdots \times 2 \times 1$ of the first n positive integers is denoted by the symbol $n!$, which is read as 'n factorial' or 'factorial n'.

> **Definition**
>
> For any non-negative integers n and k with $k \leq n$,
>
> $$\binom{n}{k} = \frac{n!}{k! \, (n - k)!}.$$
>
> This expression is called a **binomial coefficient**. It is the number of combinations of n objects taken k at a time.

For example, if we have a set with three elements, and we want to know the number of different subsets with two elements, this is given by

$$\binom{3}{2} = \frac{3!}{2! \, 1!} = 3.$$

Indeed, if we consider the set $\{a, b, c\}$, then the possible subsets with two elements are

$$\{a, b\}, \{b, c\} \text{ and } \{a, c\}.$$

Remarks

1. You may previously have met the alternative notation ${}^n C_k$ for binomial coefficients, instead of $\binom{n}{k}$, where the 'C' stands for 'combination'. Both notations are are in common use, and are sometimes read as 'n choose k'.

2. We adopt the usual convention that $0! = 1$, so that

$$\binom{n}{0} = \binom{n}{n} = 1.$$

We can now state the Binomial Theorem.

Theorem D1   Binomial Theorem

If $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$, then

$$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

$$= a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \cdots + b^n.$$

In the important special case where $a = 1$ and $b = x \in \mathbb{R}$, we have

$$(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

$$= 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots + x^n.$$

Proof   We have

$$(a + b)^n = \underbrace{(a + b) \times (a + b) \times \cdots \times (a + b)}_{n \text{ times}}.$$

When this product is multiplied out, we find that each term of the form $a^{n-k}b^k$ arises by choosing the variable a from $n - k$ of the brackets and the variable b from the remaining k brackets. Thus the coefficient of $a^{n-k}b^k$ is equal to the number of ways of choosing a subset of $n - k$ brackets (or, equivalently, a subset of k brackets) from the set of n brackets, and this is precisely $\binom{n}{k}$, as required. ■

A striking mathematical pattern appears when we expand expressions of the form $(a + b)^n$ for $n = 1, 2, \ldots$:

$(a + b)^1 = a^1 + b^1,$

$(a + b)^2 = a^2 + 2ab + b^2,$

$(a + b)^3 = (a + b)(a^2 + 2ab + b^2) = a^3 + 3a^2b + 3ab^2 + b^3,$

$(a + b)^4 = (a + b)(a^3 + 3a^2b + 3ab^2 + b^3) = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4,$

and so on.

The coefficients that appear in these expansions can be arranged as a triangular table, known as *Pascal's triangle*. The entries on the left- and right-hand edges of the triangle are all 1s, and the remaining entries can be generated by using the rule that each entry is the sum of the two nearest entries in the row above. The 1 at the top corresponds to $n = 0$ since we have $(a + b)^0 = 1$.

$$
\begin{array}{cccccccccccc}
(a+b)^0 & & & & & & 1 & & & & & \\
(a+b)^1 & & & & & 1 & & 1 & & & & \\
(a+b)^2 & & & & 1 & & 2 & & 1 & & & \\
(a+b)^3 & & & 1 & & 3 & & 3 & & 1 & & \\
(a+b)^4 & & 1 & & 4 & & 6 & & 4 & & 1 & \\
(a+b)^5 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
\vdots & & & & & & \vdots & & & & &
\end{array}
$$

Pascal's triangle is named after the French mathematician and philosopher Blaise Pascal (1623–1662). He was far from the first person to study this array of numbers, but his work on it in his *Traité du Triangle Arithmétique* was influential. Research on binomial coefficients was also carried out at about the same time by John Wallis (1616–1703) and then by Isaac Newton (1642–1727), who discovered that the Binomial Theorem can be generalised to negative and fractional powers.

Pascal's triangle had been studied centuries earlier by the Chinese mathematician Yang Hui (1238–1298) and the Persian astronomer and poet Omar Khayyam; in China, Pascal's triangle is known as the Yang Hui triangle.

Blaise Pascal

We now use the Binomial Theorem to prove some inequalities involving integers.

Worked Exercise D12

Prove that

$$2^n \geq 1 + n, \quad \text{for } n \geq 1.$$

Solution

By the Binomial Theorem, for $n \geq 1$ and $x \in \mathbb{R}$, we have

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots + x^n.$$

In the case $x \geq 0$, all the terms on the right are non-negative, so we can decrease the sum by omitting all but the first two terms.

So

$$(1 + x)^n \geq 1 + nx, \quad \text{for } x \geq 0.$$

If we now substitute $x = 1$ in this inequality, we obtain

$$2^n \geq 1 + n, \quad \text{for } n \geq 1.$$

This is the required result.

Worked Exercise D13

Prove that

$$2^{1/n} \leq 1 + \frac{1}{n}, \quad \text{for } n \geq 1.$$

Solution

We start by rearranging the required result into an equivalent form.

$$2^{1/n} \leq 1 + \frac{1}{n} \iff 2 \leq \left(1 + \frac{1}{n}\right)^n \quad \text{(by Rule 5, with } p = n\text{)}$$

The bracket on the right can now be expanded using the special case of the Binomial Theorem and, as in the last worked exercise, we can then reduce the sum by omitting all but the first two terms.

Applying the Binomial Theorem with $x = 1/n$, we get

$$\left(1 + \frac{1}{n}\right)^n = 1 + n\left(\frac{1}{n}\right) + \frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2 + \cdots + \left(\frac{1}{n}\right)^n$$

$$\geq 1 + 1 = 2.$$

Thus the inequality $2 \leq \left(1 + \frac{1}{n}\right)^n$ is true for $n \geq 1$, so it follows that the original inequality

$$2^{1/n} \leq 1 + \frac{1}{n}, \quad \text{for } n \geq 1,$$

is also true, as required.

Exercise D15

Use the Binomial Theorem to prove that

$$\left(1 + \frac{1}{n}\right)^n \geq \frac{5}{2} - \frac{1}{2n}, \quad \text{for } n \geq 1.$$

Hint: Consider the first three terms in the binomial expansion.

3.5 Mathematical induction and Bernoulli's Inequality

Another useful tool for proving inequalities involving integers is the Principle of Mathematical Induction, which you met in Unit A3.

Principle of Mathematical Induction

To prove that a statement $P(n)$ is true for $n = 1, 2, \ldots$:

1. show that $P(1)$ is true

2. show that the implication $P(k) \implies P(k+1)$ is true for $k = 1, 2, \ldots$.

Recall that the Principle can be adapted to prove that a statement $P(n)$ is true for all integers n greater than or equal to some given integer other than 1.

Worked Exercise D14

Prove that

$$2^n \geq n^2, \quad \text{for } n \geq 4.$$

Solution

Let $P(n)$ be the statement $2^n \geq n^2$.

💬 We want to prove that $P(n)$ is true for $n \geq 4$, so we start with $P(4)$. 💬

$P(4)$ is true since $2^4 = 16 = 4^2$.

Now let $k \geq 4$ and assume that $P(k)$ is true; that is,

$$2^k \geq k^2.$$

We wish to deduce that $P(k+1)$ is true; that is,

$$2^{k+1} \geq (k+1)^2.$$

Multiplying the inequality $2^k \geq k^2$ by 2 (using Rule 3) we get

$$2^{k+1} \geq 2k^2,$$

so it is sufficient for our purposes to prove that

$$2k^2 \geq (k+1)^2, \quad \text{for } k \geq 4.$$

Now,

$$2k^2 \geq (k+1)^2 \iff 2k^2 \geq k^2 + 2k + 1$$
$$\iff k^2 - 2k - 1 \geq 0 \quad \text{(by Rule 1)}$$
$$\iff (k-1)^2 - 2 \geq 0 \quad \text{(completing the square)}.$$

This last inequality certainly holds for $k \geq 4$, so we have shown that

$$2^k \geq k^2 \implies 2^{k+1} \geq (k+1)^2, \quad \text{for } k \geq 4,$$

which was our aim.

Thus

$$P(k) \implies P(k+1), \quad \text{for } k \geq 4.$$

Hence, by mathematical induction, $P(n)$ is true, for $n \geq 4$.

Exercise D16

Prove that

$$2^n \geq n^3, \quad \text{for } n \geq 10.$$

Hint: You may find it helpful to use the solution to Exercise D14.

The next inequality, called *Bernoulli's Inequality*, will be used regularly in later units. We will prove the result using mathematical induction.

Bernoulli's Inequality is named after the Swiss mathematician Jacob Bernoulli (1654–1705) who published it in his *Positiones Arithmeticae de Seriebus Infinitis* (1689), using it several times. However, it is actually due to the Walloon mathematician René-François de Sluse (1622–1685), who published it in the second edition of his *Mesolabum* (1668).

Jacob Bernoulli was the first of the remarkable Bernoulli family who, over the course of three generations, produced eight gifted mathematicians. Jacob is best known for his *Ars Conjectandi*, a pioneering book on the theory of probability which was published posthumously in 1713 by his nephew, and for his founding work on the calculus of variations.

Jacob Bernoulli

René-François de Sluse's position in the church prevented him from visiting other mathematicians, but he corresponded with many mathematicians of the day, including Blaise Pascal.

Theorem D2   Bernoulli's Inequality

If $x \in \mathbb{R}$ and $n \in \mathbb{N}$, then

$$(1+x)^n \geq 1 + nx, \quad \text{when } x \geq -1.$$

René-François de Sluse

Remarks

1. In the solution to Worked Exercise D12 you saw that $(1+x)^n \geq 1 + nx$ for $x \geq 0$ and $n \in \mathbb{N}$. Bernoulli's Inequality asserts that the same result holds under the *weaker* assumption that $x \geq -1$. Here, by a *weaker* assumption we mean an assumption that is less restrictive. Correspondingly, we say that Bernoulli's Inequality is a *stronger* result than that proved in the solution of Worked Exercise D12, because it applies more widely.

2. If we sketch the graphs of $y = (1+x)^n$ and $y = 1 + nx$, it certainly *looks* as though the first graph always lies above the second, so long as $x \geq -1$; see Figure 12. This *suggests* that Bernoulli's Inequality should hold for $x \geq -1$, but of course we need to prove it.
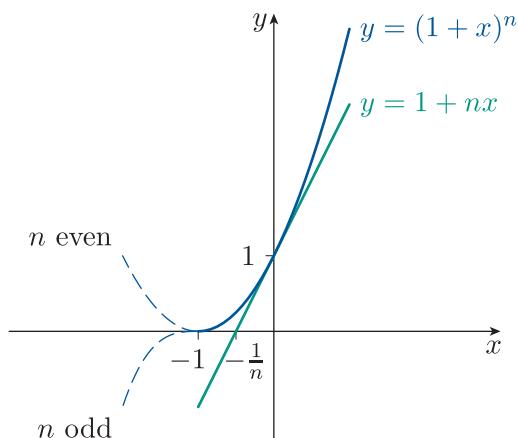
Figure 12   The graphs of $y = (1+x)^n$ and $y = 1 + nx$

Proof of Theorem D2   Let $x \geq -1$ and let $P(n)$ be the statement $(1+x)^n \geq 1 + nx$.

$P(1)$ is true since $(1+x)^1 = 1 + x$.

Now let $k \geq 1$ and assume that $P(k)$ is true; that is,

$$(1+x)^k \geq 1 + kx.$$

We wish to deduce that $P(k+1)$ is true; that is,

$$(1+x)^{k+1} \geq 1 + (k+1)x.$$

Multiplying the inequality $(1+x)^k \geq 1 + kx$ by the quantity $(1+x)$, which is non-negative because $x \geq -1$, we get

$$\begin{aligned}
(1+x)^{k+1} &\geq (1+x)(1+kx) \\
&= 1 + (k+1)x + kx^2 \\
&\geq 1 + (k+1)x,
\end{aligned}$$

since the term kx^2 is positive. Thus we have shown that

$$P(k) \implies P(k+1), \quad \text{for } k \geq 1.$$

Hence, by mathematical induction, $P(n)$ is true, for $n \geq 1$.   ∎

In Worked Exercise D13 you saw that

$$2^{1/n} \leq 1 + \frac{1}{n}, \quad \text{for } n \geq 1.$$

In the next worked exercise you will see how Bernoulli's Inequality can be used to prove another inequality involving $2^{1/n}$.

Worked Exercise D15

By applying Bernoulli's Inequality with $x = -1/(2n)$, prove that

$$2^{1/n} \geq 1 + \frac{1}{2n-1}, \quad \text{for } n \geq 1.$$

Solution

💭 We can apply Bernoulli's Inequality because $-1/(2n) \geq -1$ for $n \geq 1$. 💭

Substituting $x = -1/(2n)$ into Bernoulli's Inequality, we obtain

$$\begin{aligned}
\left(1 - \frac{1}{2n}\right)^n &\geq 1 + n\left(-\frac{1}{2n}\right) \\
&= \frac{1}{2}.
\end{aligned}$$

If we then take the nth root of both sides of this inequality (which is permissible, by Rule 5), we obtain

$$1 - \frac{1}{2n} \geq \frac{1}{2^{1/n}},$$

that is,

$$\frac{2n-1}{2n} \geq \frac{1}{2^{1/n}}$$

hence, by Rule 4,

$$2^{1/n} \geq \frac{2n}{2n-1}.$$

💬 We now write $2n = (2n-1) + 1$ to get the right-hand side into the required form, and this completes the proof. 💬

Hence,

$$2^{1/n} \geq \frac{(2n-1)+1}{2n-1} = 1 + \frac{1}{2n-1}, \quad \text{for } n \geq 1.$$

Combining the results of Worked Exercises D13 and D15, we have shown that

$$1 + \frac{1}{2n-1} \leq 2^{1/n} \leq 1 + \frac{1}{n}, \quad \text{for } n \geq 1.$$

Exercise D17

By applying Bernoulli's Inequality, first with $x = -3/(4n)$, and then with $x = 3/n$, prove that

$$1 + \frac{3}{4n-3} \leq 4^{1/n} \leq 1 + \frac{3}{n}, \quad \text{for } n \geq 1.$$

4 Least upper bounds

In this section you will meet the idea of upper and lower bounds of a set and then see how to identify the least upper bound and the greatest lower bound of a set. You will also learn about the Least Upper Bound Property of $\mathbb{R}$.

4.1 Upper bounds and lower bounds

Any finite set of real numbers has a greatest element (and a least element), but this property does not necessarily hold for sets with infinitely many elements. For example, neither of the sets $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $[0, 2)$ has a greatest element. However, the set $[0, 2)$ is *bounded above* by 2, since all points of $[0, 2)$ are less than 2.

> ### Definitions
>
> A set $E \subseteq \mathbb{R}$ is **bounded above** if there is a real number M, called an **upper bound** of E, such that
>
> $$x \leq M, \quad \text{for all } x \in E.$$
>
> If the upper bound M belongs to E, then M is called the **maximum element** of E, and is denoted by $\max E$.

Geometrically, the set E is bounded above by M if no point of E lies to the right of M on the real line.

For example, if $E = [0, 2)$, then the numbers 2, 3, 3.5 and 157.1 are all upper bounds of E, whereas the numbers 1.995, 1.5, 0 and -157.1 are not upper bounds of E. Although it may seem obvious that $[0, 2)$ has no maximum element, you may find it difficult to write down a proof of this fact. The next worked exercise demonstrates how to do this.

Worked Exercise D16

Sketch the following sets on the real line, and determine which are bounded above and which have a maximum element.

(a) $E_1 = [0, 2)$

(b) $E_2 = \{1/n : n = 1, 2, \ldots\}$

(c) $E_3 = \mathbb{N}$

> #### Solution
>
> (a)
>
>
>
> The set E_1 is bounded above. For example, $M = 2$ is an upper bound of E_1, since
>
> $$x \leq 2, \quad \text{for all } x \in E_1.$$
>
> However, E_1 has no maximum element, as we now show.
>
> The number 2 is not a maximum element, since $2 \notin E_1$. We choose a general element in E_1 and show that there is another element in E_1 which is greater.
>
> For each x in E_1 we have $x < 2$, so there is a real number y such that
>
> $$x < y < 2,$$
>
> by the Density Property of $\mathbb{R}$. Hence $y \in E_1$, so x is not a maximum element of E_1. This shows that no element of E_1 is a maximum element of E_1.

(b)
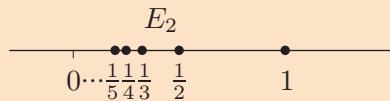
The set E_2 is bounded above. For example, $M = 1$ is an upper bound of E_2, since

$$\frac{1}{n} \le 1, \quad \text{for } n = 1, 2, \ldots.$$

Also, since $1/n = 1$ when $n = 1$, we have $1 \in E_2$ and so

$$\max E_2 = 1.$$

(c)
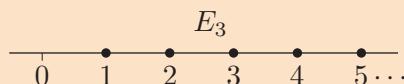
The set E_3 is not bounded above. For each real number M, there is a positive integer n such that $n > M$, by the Archimedean Property of $\mathbb{R}$. Hence M is not an upper bound of E_3.

Since E_3 is not bounded above, it has no maximum element.

Exercise D18

Sketch the following sets on the real line, and determine which are bounded above and which have a maximum element.

(a) $E_1 = (-\infty, 1]$

(b) $E_2 = \{1 - 1/n : n = 1, 2, \ldots\}$

(c) $E_3 = \{n^2 : n = 1, 2, \ldots\}$

Lower bounds are defined in a similar way to upper bounds. For example, the interval $(0, 2)$ is bounded below by 0, since

$$0 \le x, \quad \text{for all } x \in (0, 2).$$

However, 0 does not belong to $(0, 2)$, so 0 is not a minimum element of $(0, 2)$. In fact, $(0, 2)$ has no minimum element.

> **Definitions**
>
> A set $E \subseteq \mathbb{R}$ is **bounded below** if there is a real number m, called a **lower bound** of E, such that
>
> $$m \le x, \quad \text{for all } x \in E.$$
>
> If the lower bound m belongs to E, then m is called the **minimum element** of E, and is denoted by $\min E$.

Geometrically, the set E is bounded below by m if no point of E lies to the left of m on the real line.

Exercise D19

Determine which of the following sets are bounded below and which have a minimum element. (The sketches you made in Exercise D18 may help.)

(a) $E_1 = (-\infty, 1]$

(b) $E_2 = \{1 - 1/n : n = 1, 2, \ldots\}$

(c) $E_3 = \{n^2 : n = 1, 2, \ldots\}$

The following terminology is also useful.

> **Definitions**
>
> A set $E \subseteq \mathbb{R}$ is **bounded** if E is bounded above and bounded below; the set E is **unbounded** if it is not bounded.

For example, of the sets you met in Exercises D18 and D19, the set $E_2 = \{1 - 1/n : n = 1, 2, \ldots\}$ is bounded, but $E_1 = (-\infty, 1]$ and $E_3 = \{n^2 : n = 1, 2, \ldots\}$ are unbounded.

4.2   Least upper bounds and greatest lower bounds

We have seen that the set $[0, 2)$ has no maximum element. However, $[0, 2)$ has many upper bounds, for example $2, 3, 3.5$ and 157.1. Among all these upper bounds, the number 2 is the *least* upper bound, because any number less than 2 is not an upper bound of $[0, 2)$, as illustrated in Figure 13.
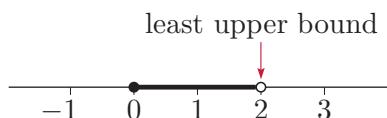

Figure 13   The least upper bound of $[0, 2)$

The least upper bound of a set is also called the *supremum* of a set. This comes from the Latin word *supremus* meaning 'highest'.

> ### Definition
>
> A real number M is the **least upper bound**, or **supremum**, of a set $E \subseteq \mathbb{R}$ if
>
> 1. M is an upper bound of E
>
> 2. each number $M' < M$ is not an upper bound of E.
>
> In this case, we write $M = \sup E$.

If E has a maximum element, $\max E$, then $\sup E = \max E$. For example, the closed interval $[0, 2]$ has maximum element 2, so it has least upper bound 2.

If a set does not have a maximum element but is bounded above, then we may be able to guess the value of its least upper bound. As in the example $E = [0, 2)$, there may be an obvious 'missing point' at the upper end of the set. We now see how to *prove* that our guess is correct.

Worked Exercise D17

Prove that the least upper bound of $[0, 2)$ is 2.

> ### Solution
>
> 🔍 First check that 2 is *an* upper bound. 💬
>
> We know that $M = 2$ is an upper bound of $[0, 2)$ because
>
> $$x \le 2, \quad \text{for all } x \in [0, 2).$$
>
> 🔍 To show that 2 is the *least* upper bound, we must prove that each number $M' < 2$ is *not* an upper bound of $[0, 2)$. 💬
>
> Suppose that $M' < 2$. We must find an element x in $[0, 2)$ which is greater than M'. By the Density Property, there is a real number x such that
>
> $$M' < x < 2 \quad \text{and} \quad x \ge 0.$$
>
> Thus $x \in [0, 2)$ and $x > M'$, which shows that M' is not an upper bound of $[0, 2)$. Hence $M = 2$ is the least upper bound of $[0, 2)$.

The solution to Worked Exercise D17 illustrates the following strategy for determining the least upper bound of a set, if there is one.

> ### Strategy D1
>
> To show that M is the least upper bound (supremum) of a subset E of $\mathbb{R}$, check that:
>
> 1. $x \leq M$, for *all* $x \in E$
> 2. if $M' < M$, then there is *some* $x \in E$ such that $x > M'$.
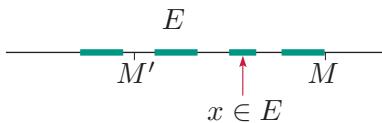
Figure 14   The points and sets in Strategy D1

The points and sets involved in Strategy D1 are illustrated in Figure 14, with the set E being indicated by bold green lines. If M is an upper bound of a set E and $M \in E$, then steps 1 and 2 of the strategy are automatically satisfied, so $M = \sup E = \max E$.

Worked Exercise D18

Determine the least upper bound of the set

$$E = \{1 - 1/n^2 : n = 1, 2, \ldots\}.$$

> #### Solution
>
> 🗨 We first guess the least upper bound of E. A sketch may help. 💭
>
>
> 🗨 We guess from the sketch that the least upper bound of E is $M = 1$. We now use Strategy D1. 💭
>
> We have that 1 is an upper bound of E, since
>
> $$1 - \frac{1}{n^2} \leq 1, \quad \text{for } n = 1, 2, \ldots.$$
>
> We now need to show that, if $M' < 1$, then M' is not an upper bound of E; that is, there is some natural number n such that
>
> $$1 - \frac{1}{n^2} > M'.$$
>
> 🗨 We show this by rearranging the inequality into an equivalent form with just n on one side. 💭
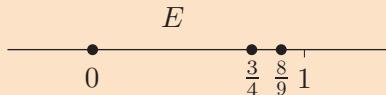
Suppose that $0 < M' < 1$. We have

$$1 - \frac{1}{n^2} > M' \iff 1 - M' > \frac{1}{n^2}$$

$$\iff \frac{1}{1 - M'} < n^2$$

$$\iff \sqrt{\frac{1}{1 - M'}} < n \quad (\text{since } (1 - M') > 0 \text{ and } n > 0).$$

We can certainly choose n so that this final inequality holds, by the Archimedean Property of $\mathbb{R}$. Hence, for this value of n, we have

$$1 - \frac{1}{n^2} > M',$$

so M' is not an upper bound of E.

It follows that 1 is the least upper bound of E.

Exercise D20

Determine the least upper bound, if it exists, for each of the following sets. (You will find it helpful to refer to your solution to Exercise D18.)

(a) $E_1 = (-\infty, 1]$

(b) $E_2 = \{1 - 1/n : n = 1, 2, \ldots\}$

(c) $E_3 = \{n^2 : n = 1, 2, \ldots\}$

The *greatest lower bound* or *infimum* of a set is defined in a similar way to the least upper bound. The word infimum comes from the Latin word *infimus* meaning 'least'.

Definition

A real number m is the **greatest lower bound**, or **infimum**, of a set $E \subseteq \mathbb{R}$ if

1. m is a lower bound of E

2. each number $m' > m$ is not a lower bound of E.

In this case, we write $m = \inf E$.

If E has a minimum element, then $\inf E = \min E$. For example, the closed interval $[0, 2]$ has minimum element 0, so it has greatest lower bound 0.

The following strategy for proving that a number is the greatest lower bound of a set is similar to Strategy D1.

> ### Strategy D2
>
> To show that m is the greatest lower bound (infimum) of a subset E of $\mathbb{R}$, check that:
>
> 1. $x \geq m$, for *all* $x \in E$
>
> 2. if $m' > m$, then there is *some* $x \in E$ such that $x < m'$.
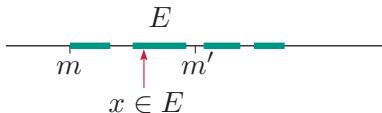
Figure 15   The sets and points in Strategy D2

The sets and points involved in Strategy D2 are illustrated in Figure 15, with the set E being indicated by bold lines. If m is a lower bound of E and $m \in E$, then steps 1 and 2 of the strategy are automatically satisfied, so $m = \inf E = \min E$.

> ### Exercise D21
>
> Determine the greatest lower bound, if it exists, for each of the following sets.
>
> (a) $E_1 = (1, 5]$ (b) $E_2 = \{1/n^2 : n = 1, 2, \ldots\}$

4.3   Least Upper Bound Property

In the exercises and worked exercises in the previous subsection it was straightforward to guess the values of $\sup E$ and $\inf E$. Sometimes, however, this is not the case. For example, if

$$E = \{(1 + 1/n)^n : n = 1, 2, \ldots\} = \left\{ \left(\tfrac{2}{1}\right)^1, \left(\tfrac{3}{2}\right)^2, \left(\tfrac{4}{3}\right)^3, \ldots \right\},$$

then it is not very easy to guess the value of the least upper bound of E. It turns out that

$$\sup E = e = 2.718\,28\ldots.$$

In cases like this it can be reassuring to know that $\sup E$ does exist, even though it may be difficult to find.

The existence of $\sup E$ is guaranteed by the following fundamental result, which is the basis for many other results in analysis. (This is an example of an *existence theorem*, that is, a theorem that asserts that a mathematical object, such as a real number with a certain property, must exist, but does not tell you what it is.)

> ### Least Upper Bound Property of $\mathbb{R}$
>
> Let E be a non-empty subset of $\mathbb{R}$. If E is bounded above, then E has a least upper bound.

This property of $\mathbb{R}$ is very plausible geometrically. If the set E lies entirely to the left of some number M, then you can imagine decreasing the value of M steadily until any further decrease causes M to be less than some point of E. At this point, $\sup E$ has been reached, as shown in Figure 16, where the set E is indicated by bold green lines.

$$E$$
$$\sup E \qquad M$$
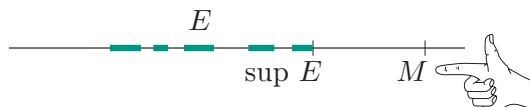
Figure 16   The Least Upper Bound Property of $\mathbb{R}$

The Least Upper Bound Property of $\mathbb{R}$ can be used to show that $\mathbb{R}$ does indeed include non-recurring decimals which represent irrational numbers such as $\sqrt{2}$, as was claimed in Section 1. It can also be used to define the arithmetic operations of addition and multiplication with such decimals. We discuss this further in Section 5.

As you would expect, there is a corresponding property for lower bounds.

Greatest Lower Bound Property of $\mathbb{R}$

Let E be a non-empty subset of $\mathbb{R}$. If E is bounded below, then E has a greatest lower bound.

We now give a proof of the Least Upper Bound Property in the case when the set E contains at least one positive number. The proof in the general case can be reduced to this special case, but we do not give the details. Try to follow the proof if you are interested, but you can omit it if you are short of time.

Proof of the Least Upper Bound Property   Let E be a subset of $\mathbb{R}$ that is bounded above and contains at least one positive number.

We first choose a suitable candidate for the least upper bound, and then use Strategy D1.

We apply the following procedure to give us the successive digits in a decimal $a_0.a_1a_2\ldots$, which we then prove to be the least upper bound of E.

We choose in succession:

the greatest integer a_0 such that a_0 is not an upper bound of E

the greatest digit a_1 such that $a_0.a_1$ is not an upper bound of E

the greatest digit a_2 such that $a_0.a_1a_2$ is not an upper bound of E

$$\vdots$$

the greatest digit a_n such that $a_0.a_1\ldots a_n$ is not an upper bound of E

$$\vdots$$

Thus at the nth stage, the selected digit a_n has the properties that

$$a_0.a_1a_2\ldots a_n \text{ is not an upper bound of } E \tag{4}$$

$$a_0.a_1a_2\ldots a_n + \frac{1}{10^n} \text{ is an upper bound of } E. \tag{5}$$

Since E contains at least one positive number, the numbers in (4) and (5) are positive rationals.

We now use Strategy D1 to prove that the least upper bound of E is the non-terminating decimal

$$a = a_0.a_1a_2\ldots.$$

First we have to show that a is an upper bound of E; that is, if $x \in E$, then $x \le a$. To do this, we prove the equivalent contrapositive statement: if $x > a$, then $x \notin E$.

Note that our procedure for choosing a always produces a non-terminating decimal. For example, it would give $a = 1.49999\ldots$, rather than $a = 1.5$. For the purpose of making comparisons, we therefore represent x in the same way.

Let $x > a$, and represent x as a non-terminating decimal $x = x_0.x_1x_2\ldots$.

Since $x > a$, there is a least integer n such that $x_n > a_n$, which means that $x_n \ge a_n + 1$. Thus for this value of n we have

$$x_0.x_1x_2\ldots x_n \ge a_0.a_1a_2\ldots a_n + \frac{1}{10^n} > a_0.a_1a_2\ldots,$$

so $x_0.x_1x_2\ldots x_n$ is an upper bound of E, by statement (5). Since x is non-terminating, $x > x_0.x_1x_2\ldots x_n$, so $x \notin E$, as required.

To complete the proof we have to show that if $a' < a$, then a' is not an upper bound of E. Let $a' < a$. Then there is an integer n such that

$$a' < a_0.a_1a_2\ldots a_n,$$

so a' is not an upper bound of E, by statement (4).

Thus we have proved that a is the least upper bound of E. ■

5   Manipulating real numbers

This section is intended for reading only. There are no exercises on this section.

5.1   Arithmetic with real numbers

At the end of Section 1 we discussed the non-recurring decimals representing the irrationals $\sqrt{2}$ and π, which begin

$$\sqrt{2} = 1.414\,213\,56\ldots \quad \text{and} \quad \pi = 3.141\,592\,65\ldots,$$

and asked whether it is possible to add and multiply these numbers to obtain another real number. We now explain how this can be done using the Least Upper Bound Property of $\mathbb{R}$.

A natural way to obtain a sequence of approximations to the sum $\sqrt{2} + \pi$ is to truncate each of the above decimals and then form the sums of these truncated decimals. If each of the decimals is truncated at the same decimal place, then we obtain the following sequences of approximations, which are increasing.

$\sqrt{2}$	π	$\sqrt{2}+\pi$
1	3	4
1.4	3.1	4.5
1.41	3.14	4.55
1.414	3.141	4.555
1.4142	3.1415	4.5557
$\vdots$	$\vdots$	$\vdots$

Intuitively we expect that the sum $\sqrt{2} + \pi$ is greater than each of the numbers in the right-hand column, but 'only just'. To accord with our intuition, therefore, we *define* the sum $\sqrt{2} + \pi$ to be the least upper bound of the set of numbers in the right-hand column; that is,

$$\sqrt{2} + \pi = \sup\{4, 4.5, 4.55, 4.555, 4.5557, \ldots\}.$$

To be sure that this definition makes sense, we need to show that this set is bounded above. But all the truncations of $\sqrt{2}$ are less than 1.5 and all those of π are less than 4. Hence, all the sums in the right-hand column are less than $1.5 + 4 = 5.5$. So, by the Least Upper Bound Property of $\mathbb{R}$, the set of numbers in the right-hand column *does* have a least upper bound and we *can* define $\sqrt{2} + \pi$ this way.

This method can be used to define the sum of any pair of positive real numbers.

Let us check that this method of adding decimals gives the correct answer when we use it in a familiar case. Consider the simple calculation

$$\tfrac{1}{3} + \tfrac{2}{3} = 0.333\ldots + 0.666\ldots.$$

Truncating each of these decimals and forming the sums, we obtain the set

$$\{0, 0.9, 0.99, 0.999, \ldots\}.$$

The supremum of this set is $0.999\ldots = 1$, which is the correct answer. (We are not suggesting that this is a practical method for adding rationals!)

We can define the product of any two positive real numbers in a similar way. For example, to define $\sqrt{2} \times \pi$ we can form the sequence of products of their truncations.

$\sqrt{2}$	π	$\sqrt{2} \times \pi$
1	3	3
1.4	3.1	4.34
1.41	3.14	4.4274
1.414	3.141	4.441374
1.4142	3.1415	4.4427093
$\vdots$	$\vdots$	$\vdots$

As before, we define $\sqrt{2} \times \pi$ to be the least upper bound of the set of numbers in the right-hand column.

Similar ideas can be used to define the operations of subtraction and division, but we omit the details.

In this way we can define arithmetic with real numbers in terms of the familiar arithmetic with rationals by using the Least Upper Bound Property of $\mathbb{R}$. Moreover, it can be proved that these operations in $\mathbb{R}$ satisfy all the usual properties of a field; you saw these listed in Subsection 1.5.

5.2   Existence of roots

Just as we usually take for granted the basic arithmetic operations with real numbers, so we usually assume that, given any positive real number a, there is a unique positive real number $b = \sqrt{a}$ such that $b^2 = a$. We now discuss the justification for this assumption.

First, here is a geometric justification. Given line segments of lengths 1 and a, we can construct a semicircle with diameter $a + 1$ as shown in Figure 17.
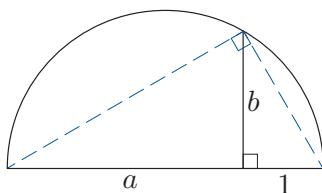

Figure 17   A semicircle with diameter $a + 1$

Using similar triangles, we find that

$$\frac{a}{b} = \frac{b}{1}, \quad \text{so} \quad b^2 = a.$$

This shows that there should be a positive real number b such that $b^2 = a$, in order that the length of the vertical line segment in this figure can be described exactly. But does $b = \sqrt{a}$ always exist *exactly* as a real number? In fact it does, and a more general result is true.

> ## Theorem D3
>
> For each positive real number a and each integer $n > 1$, there is a unique positive real number b such that
>
> $$b^n = a.$$

We call this positive number b the **nth root** of a, and write $b = \sqrt[n]{a}$. We also define $\sqrt[n]{0} = 0$, since $0^n = 0$.

In the special case $a = 2$ and $n = 2$, Theorem D3 asserts the existence of a positive real number b such that

$$b^2 = 2.$$

Here is a direct proof of Theorem D3 in this special case. (A proof of the general case is given in Section 4 of Unit D4.) We choose the numbers $1, 1.4, 1.41, 1.414, \ldots$ to satisfy the inequalities:

$$\left.\begin{array}{c} 1^2 < 2 < 2^2 \\ (1.4)^2 < 2 < (1.5)^2 \\ (1.41)^2 < 2 < (1.42)^2 \\ (1.414)^2 < 2 < (1.415)^2 \\ \vdots \end{array}\right\} \tag{6}$$

This process gives an infinite decimal $b = 1.414\ldots$ and we claim that

$$b^2 = (1.414\ldots)^2 = 2.$$

This can be proved using our method of multiplying decimals.

b	b	b^2
1	1	1
1.4	1.4	1.96
1.41	1.41	1.9881
1.414	1.414	1.999396
$\vdots$	$\vdots$	$\vdots$

We have to prove that the least upper bound of the set E of numbers in the right-hand column is 2. In other words,

$$\sup E = \sup\{1, 1.4^2, 1.41^2, 1.414^2, \ldots\} = 2.$$

To do this, we use Strategy D1.

First we check that $M = 2$ is an upper bound of E. This follows from the left-hand inequalities in (6).

Next we check that if $M' < 2$, then there is a number in E which is greater than M'. To prove this, we put

$$x_0 = 1, \quad x_1 = 1.4, \quad x_2 = 1.41, \quad x_3 = 1.414, \quad \ldots.$$

By the right-hand inequalities in (6) we have, for $n = 0, 1, 2, \ldots$,

$$2 < \left(x_n + \frac{1}{10^n} \right)^2 = x_n^2 + \frac{2x_n}{10^n} + \left(\frac{1}{10^n} \right)^2,$$

so

$$x_n^2 > 2 - \frac{1}{10^n} \left(2x_n + \frac{1}{10^n} \right).$$

Since $x_n < 2$, we have

$$2x_n + \frac{1}{10^n} < 2 \times 2 + 1 = 5$$

and so

$$x_n^2 > 2 - \frac{5}{10^n} = 1.\underbrace{99\ldots95}_{n \text{ digits}}.$$

Thus if $M' < 2$, then we *can* choose n so large that $x_n^2 > M'$. This proves that the least upper bound of the set E is 2, so

$$b^2 = (1.414\ldots)^2 = 2,$$

as claimed. Thus $b = 1.414\ldots$ is a decimal representation of $\sqrt{2}$.

5.3   Powers

Having discussed nth roots, we are now in a position to define the expression a^x, where a is positive and x is a rational power (or exponent).

> **Definition**
>
> If $a > 0$, $m \in \mathbb{Z}$ and $n \in \mathbb{N}$, then
>
> $$a^{m/n} = (\sqrt[n]{a})^m$$
>
> or, equivalently,
>
> $$a^{m/n} = \sqrt[n]{a^m}.$$

For example, for $a > 0$ with $m = 1$ we have $a^{1/n} = \sqrt[n]{a}$, and with $m = 2$ and $n = 3$ we have $a^{2/3} = (\sqrt[3]{a})^2$.

This notation is particularly useful because rational powers satisfy the following laws, whose proofs depend on Theorem D3.

Index Laws

If $a, b > 0$ and $x, y \in \mathbb{Q}$, then

$$a^x b^x = (ab)^x, \quad a^x a^y = a^{x+y} \quad \text{and} \quad (a^x)^y = a^{xy}.$$

Remarks

1. If x and y are *integers*, then these laws also hold for all non-zero real numbers a and b, not just positive ones. However, if x and y are not integers, then we must have $a > 0$ and $b > 0$. For example, $(-1)^{1/2}$ is not defined as a real number.

2. Each positive number has *two* nth roots when n is even. For example, $2^2 = (-2)^2 = 4$. We adopt the convention that, for $a > 0$, $\sqrt[n]{a}$ and $a^{1/n}$ always denote the *positive* nth root of a. If we wish to refer to both roots (for example, when solving equations), then we write $\pm\sqrt[n]{a}$.

We conclude this section by briefly discussing the meaning of a^x when $a > 0$ and x is an arbitrary real number. We have defined a^x when x is rational, but the same definition does not work if x is irrational. However, it is common practice to write expressions such as

$$\sqrt{2}^{\sqrt{2}}$$

and even to apply the Index Laws to give equalities such as

$$\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2} \times \sqrt{2}} = (\sqrt{2})^2 = 2.$$

Such manipulations *can* be justified, and by the end of this book you will have seen one way to do this. Moreover, the justification uses several key ideas from the book, including convergence of sequences, convergence of series and continuity.

Summary

In this unit you have met many of the basic ideas and techniques you will need as you continue your study of analysis.

You have seen how the real numbers and their arithmetic operations can be precisely defined using infinite decimals. You have learned how to manipulate, solve and prove inequalities, and met the Triangle Inequality and Bernoulli's Inequality, both of which are widely used in analysis. You have studied upper and lower bounds of sets of real numbers, and discovered how to show whether a given number is a least upper bound or a greatest lower bound of such a set.

Finally, you have seen that whilst both the rationals and the reals are ordered fields, only the real numbers possess the Least Upper Bound Property, namely that every non-empty subset of $\mathbb{R}$ that is bounded above has a least upper bound. This is the key property that distinguishes the reals from the rationals, and it underlies many of the ideas you will meet in future analysis units.

Learning outcomes

After working through this unit, you should be able to:

- explain the relationship between *rational numbers* and *recurring decimals*

- explain the term *irrational number*

- understand how the *real numbers* can be defined in terms of infinite decimals

- find a rational and an irrational number between any two distinct real numbers

- solve inequalities by rearranging them into simpler equivalent forms

- solve inequalities involving modulus signs

- use the Triangle Inequality

- use the Binomial Theorem, mathematical induction and Bernoulli's Inequality to prove inequalities which involve an integer n

- explain the terms *bounded above*, *bounded below* and *bounded*

- use the strategies for determining least upper bounds and greatest lower bounds

- state the Least Upper Bound Property of $\mathbb{R}$ and the Greatest Lower Bound Property of $\mathbb{R}$

- explain how the Least Upper Bound Property is used to define arithmetic operations with real numbers

- explain the meaning of rational powers.

Solutions to exercises

Solution to Exercise D1

Since $45 \times 20 = 900$ and $17 \times 53 = 901$, we have $45/53 < 17/20$. Thus

$$-1 < -\frac{17}{20} < -\frac{45}{53} < 0 < \frac{45}{53} < \frac{17}{20} < 1.$$

Solution to Exercise D2

Let a, b be distinct rationals, where $a < b$.

Let $c = \frac{1}{2}(a + b)$; then c is rational, and
$$c - a = \tfrac{1}{2}(b - a) > 0,$$
$$b - c = \tfrac{1}{2}(b - a) > 0,$$
so $a < c < b$.

Solution to Exercise D3

We have $\frac{1}{7} = 0.142\,857\,142\,857\ldots$.

Solution to Exercise D4

(a) Let $x = 0.\overline{231}$.

Multiplying both sides by 10^3, we obtain
$$1000x = 231.\overline{231} = 231 + x.$$

Hence
$$999x = 231, \quad \text{so} \quad x = \frac{231}{999} = \frac{77}{333}.$$

(b) Let $x = 0.\overline{81}$.

Multiplying both sides by 10^2, we obtain
$$100x = 81.\overline{81} = 81 + x.$$

Hence
$$99x = 81, \quad \text{so} \quad x = \frac{81}{99} = \frac{9}{11}.$$

Thus
$$2.2\overline{81} = 2 + \frac{2}{10} + \frac{9}{110} = \frac{251}{110}.$$

Solution to Exercise D5

$\frac{17}{20} = 0.85$ and $\frac{45}{53} = 0.84\ldots$, so $\frac{45}{53} < \frac{17}{20}$.

Solution to Exercise D6

$$x = 0.34 \quad \text{and} \quad y = 0.340\,010\,010\,001\ldots,$$

where $010\,010\,001\ldots$ is a non-recurring tail.

Solution to Exercise D7

(a) Rule 2 with $c = 3$ gives $x + 3 > 5$.

(b) Rule 1 followed by Rule 3 with $c = -1$ gives $2 - x < 0$.

(c) Rule 3 with $c = 5$ followed by Rule 2 with $c = 2$ gives $5x + 2 > 12$.

(d) Part (c) followed by Rule 4 gives $1/(5x + 2) < 1/12$.

Solution to Exercise D8

(a) Note that this inequality cannot be solved by cross-multiplying, because $x^2 - 1$ can be positive, zero or negative, depending on the value of x. Rearranging the inequality (using Rule 1 and Rule 3 with $c = -4$), we obtain

$$\frac{4x - x^2 - 7}{x^2 - 1} \geq 3 \iff \frac{4x - x^2 - 7}{x^2 - 1} - 3 \geq 0$$
$$\iff \frac{4x - 4x^2 - 4}{x^2 - 1} \geq 0$$
$$\iff \frac{x^2 - x + 1}{x^2 - 1} \leq 0$$
$$\iff \frac{\left(x - \frac{1}{2}\right)^2 + \frac{3}{4}}{x^2 - 1} \leq 0.$$

Since $\left(x - \frac{1}{2}\right)^2 + \frac{3}{4} > 0$, for all x, the inequality holds if and only if $x^2 - 1 = (x - 1)(x + 1) < 0$. (The fraction is undefined when $x^2 - 1 = 0$.)

Hence the solution set is
$$\left\{ x : \frac{4x - x^2 - 7}{x^2 - 1} \geq 3 \right\} = (-1, 1).$$

(b) Rearranging the inequality (using Rule 1), we obtain
$$2x^2 \geq (x + 1)^2 \iff 2x^2 \geq x^2 + 2x + 1$$
$$\iff x^2 - 2x - 1 \geq 0$$
$$\iff (x - 1)^2 - 2 \geq 0$$
$$\iff (x - 1)^2 \geq 2.$$

Hence the solution set is

$$\{x : 2x^2 \geq (x+1)^2\}$$
$$= \{x : x - 1 \leq -\sqrt{2}\} \cup \{x : x - 1 \geq \sqrt{2}\}$$
$$= (-\infty, 1 - \sqrt{2}] \cup [1 + \sqrt{2}, \infty).$$

Solution to Exercise D9

The expression $\sqrt{2x^2 - 2}$ is defined, and non-negative, when $2x^2 - 2 \geq 0$, that is, for $x^2 \geq 1$. Thus $\sqrt{2x^2 - 2}$ is defined if x lies in $(-\infty, -1] \cup [1, \infty)$.

For $x \geq 1$, using Rule 5 with $p = 2$, we see that

$$\sqrt{2x^2 - 2} > x \iff 2x^2 - 2 > x^2$$
$$\iff x^2 > 2.$$

So the part of the solution set in $[1, \infty)$ is $(\sqrt{2}, \infty)$.

For $x \leq -1$,

$$\sqrt{2x^2 - 2} \geq 0 > x,$$

so the whole of $(-\infty, -1]$ lies in the solution set.

Hence the complete solution set is

$$\{x : \sqrt{2x^2 - 2} > x\} = (-\infty, -1] \cup (\sqrt{2}, \infty).$$

Solution to Exercise D10

(a) Rearranging the inequality (using Rule 6 and Rule 5 with $p = 2$), we obtain

$$|2x^2 - 13| < 5 \iff -5 < 2x^2 - 13 < 5$$
$$\iff 8 < 2x^2 < 18$$
$$\iff 4 < x^2 < 9$$
$$\iff 4 < |x|^2 < 9$$
$$\iff 2 < |x| < 3.$$

Hence the solution set is
$$\{x : |2x^2 - 13| < 5\} = (-3, -2) \cup (2, 3).$$

(b) Rearranging the inequality (using Rule 5 with $p = 2$), we have

$$|x - 1| \leq 2|x + 1| \iff (x - 1)^2 \leq 4(x + 1)^2$$
$$\iff x^2 - 2x + 1 \leq 4x^2 + 8x + 4$$
$$\iff 0 \leq 3x^2 + 10x + 3$$
$$\iff 0 \leq (3x + 1)(x + 3).$$

Hence the solution set is
$$\{x : |x - 1| \leq 2|x + 1|\} = (-\infty, -3] \cup [-\tfrac{1}{3}, \infty).$$

Solution to Exercise D11

(a) Suppose that $|a| \leq \frac{1}{2}$.

The Triangle Inequality gives

$$|a + 1| \leq |a| + |1|$$
$$\leq \tfrac{1}{2} + 1 \quad \left(\text{since } |a| \leq \tfrac{1}{2}\right)$$
$$= \tfrac{3}{2}.$$

Hence

$$|a| \leq \tfrac{1}{2} \implies |a + 1| \leq \tfrac{3}{2}.$$

(b) Suppose that $|b| < \frac{1}{2}$.

The backwards form of the Triangle Inequality gives

$$|b^3 - 1| \geq \left||b^3| - |1|\right|$$
$$= \left||b|^3 - 1\right|$$
$$\geq 1 - |b|^3.$$

Now

$$|b| < \tfrac{1}{2} \implies |b|^3 < \tfrac{1}{8} \implies 1 - |b|^3 > \tfrac{7}{8},$$

so, from the previous chain of inequalities,

$$|b| < \tfrac{1}{2} \implies |b^3 - 1| > \tfrac{7}{8}.$$

Solution to Exercise D12

Rearranging the inequality (using Rule 2 and Rule 3), we obtain

$$\frac{1}{2}\left(a + \frac{2}{a}\right) < a \iff \frac{1}{2}a + \frac{1}{a} < a$$
$$\iff \frac{1}{a} < \frac{1}{2}a$$
$$\iff 2 < a^2.$$

Since the final inequality is true by assumption, the first inequality must also be true. Hence

$$\frac{1}{2}\left(a + \frac{2}{a}\right) < a, \quad \text{if } a > 0 \text{ and } a^2 > 2.$$

Solution to Exercise D13

Rearranging the inequality (using Rule 3 and Rule 1), we obtain

$$\frac{3n}{n^2 + 2} < 1 \iff 3n < n^2 + 2$$
$$\iff 0 < n^2 - 3n + 2$$
$$\iff 0 < (n-1)(n-2),$$

and this final inequality certainly holds for $n > 2$. So

$$\frac{3n}{n^2 + 2} < 1, \quad \text{for } n > 2.$$

Solution to Exercise D14

Using Rule 3 for rearranging the inequalities, we obtain

$$2n^3 \geq (n+1)^3 \iff 2 \geq \left(\frac{n+1}{n}\right)^3.$$

The inequality on the right is certainly true for $n = 4$, since

$$\left(\frac{5}{4}\right)^3 = \frac{125}{64} < 2,$$

and, in fact, for $n \geq 4$ we have

$$\left(\frac{n+1}{n}\right)^3 = \left(1 + \frac{1}{n}\right)^3 \leq \left(1 + \frac{1}{4}\right)^3 = \left(\frac{5}{4}\right)^3,$$

so that

$$\left(\frac{n+1}{n}\right)^3 \leq 2.$$

Hence

$$2n^3 \geq (n+1)^3, \quad \text{for } n \geq 4.$$

Solution to Exercise D15

We substitute $x = 1/n$ in the Binomial Theorem for $(1+x)^n$, and notice that all the terms are positive since $n \geq 1$; this gives

$$\left(1 + \frac{1}{n}\right)^n = 1 + n\left(\frac{1}{n}\right) + \frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2$$
$$+ \cdots + \left(\frac{1}{n}\right)^n$$
$$\geq 1 + 1 + \frac{n-1}{2n}$$
$$= 2 + \frac{1}{2} - \frac{1}{2n} = \frac{5}{2} - \frac{1}{2n}.$$

Hence

$$\left(1 + \frac{1}{n}\right)^n \geq \frac{5}{2} - \frac{1}{2n}, \quad \text{for } n \geq 1.$$

Solution to Exercise D16

Let $P(n)$ be the statement $2^n \geq n^3$.

$P(10)$ is true since $2^{10} = 1024 > 10^3$.

Now let $k \geq 10$ and assume that $P(k)$ is true; that is,

$$2^k \geq k^3.$$

We wish to deduce that $P(k+1)$ is true; that is,

$$2^{k+1} \geq (k+1)^3.$$

Multiplying the inequality $2^k \geq k^3$ by 2 we get

$$2^{k+1} \geq 2k^3,$$

so it is sufficient for our purposes to prove that

$$2k^3 \geq (k+1)^3, \quad \text{for } k \geq 10.$$

This inequality is true, by Exercise D14. (It holds in fact for $k \geq 4$.) Hence

$$2^{k+1} \geq (k+1)^3, \quad \text{for } k \geq 10.$$

Thus

$$P(k) \implies P(k+1), \quad \text{for } k \geq 10.$$

Hence, by mathematical induction, $P(n)$ is true, for $n \geq 10$.

Solution to Exercise D17

Applying Bernoulli's Inequality with $x = -3/(4n)$, we obtain

$$\left(1 - \frac{3}{4n}\right)^n \geq 1 + n\left(-\frac{3}{4n}\right) = \frac{1}{4}.$$

Hence, by Rule 5 with $p = n$, we get

$$1 - \frac{3}{4n} \geq \frac{1}{4^{1/n}}, \quad \text{for } n \geq 1.$$

Using Rule 4, we can rewrite this inequality in the form

$$\frac{4n}{4n - 3} \leq 4^{1/n}, \quad \text{for } n \geq 1,$$

so that

$$\frac{4n - 3 + 3}{4n - 3} = 1 + \frac{3}{4n - 3} \leq 4^{1/n}, \quad \text{for } n \geq 1,$$

as required.

Next, applying Bernoulli's Inequality with $x = 3/n$, we obtain

$$\left(1 + \frac{3}{n}\right)^n \geq 1 + n\left(\frac{3}{n}\right) = 4.$$

Hence, using Rule 5 with $p = n$, we get

$$1 + \frac{3}{n} \geq 4^{1/n}, \quad \text{for } n \geq 1,$$

as required.

Putting these two results together, we get the required inequality.

Solution to Exercise D18

(a)
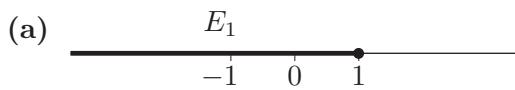
The set E_1 is bounded above. For example, $M = 1$ is an upper bound of E_1, since

$$x \leq 1, \quad \text{for all } x \in E_1.$$

Also, $\max E_1 = 1$, since $1 \in E_1$.

(b)
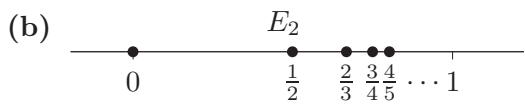
The set E_2 is bounded above. For example, $M = 1$ is an upper bound of E_2, since

$$1 - \frac{1}{n} \leq 1, \quad \text{for } n = 1, 2, \ldots.$$

However, E_2 has no maximum element. For each $x \in E_2$ we have $x = 1 - 1/n$, for some $n \in \mathbb{N}$, so there is another element of E_2 for example $1 - 1/(n+1)$, such that

$$1 - \frac{1}{n} < 1 - \frac{1}{n+1} \quad \left(\text{since } \frac{1}{n+1} < \frac{1}{n}\right).$$

Hence x is not a maximum element of E_2.
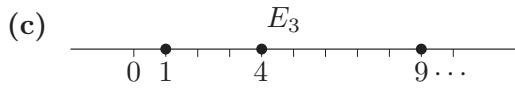
(c)

The set E_3 is not bounded above. For each real number M, there is a positive integer n such that $n^2 > M$ (for instance, take $n > M$, which implies that $n^2 \geq n > M$). Hence M is not an upper bound of E_3.

It follows that E_3 cannot have a maximum element.

Solution to Exercise D19

(a) The set $E_1 = (-\infty, 1]$ is not bounded below. For each real number m, there is a negative real number x such that $x < m$. Since $x \in E_1$, the number m is not a lower bound of E_1.

It follows that E_1 cannot have a minimum element.

(b) The set E_2 is bounded below by 0, since

$$1 - \frac{1}{n} \geq 0, \quad \text{for } n = 1, 2, \ldots.$$

Also, $0 \in E_2$, so $\min E_2 = 0$.

(c) The set E_3 is bounded below by 1, since

$$n^2 \geq 1, \quad \text{for } n = 1, 2, \ldots.$$

Also, $1 \in E_3$, so $\min E_3 = 1$.

Solution to Exercise D20

(a) The set $E_1 = (-\infty, 1]$ has maximum element 1, so

$$\sup E_1 = \max E_1 = 1.$$

(b) We know from Exercise D18 that 1 is an upper bound of E_2, since

$$1 - \frac{1}{n} \leq 1, \quad \text{for } n = 1, 2, \ldots.$$

To show that $M = 1$ is the least upper bound of E_2, we have to prove that, if $M' < 1$, then there is an element $1 - 1/n$ of E_2 such that

$$1 - \frac{1}{n} > M'.$$

Suppose that $0 < M' < 1$. We have

$$1 - \frac{1}{n} > M'$$
$$\iff \quad 1 - M' > \frac{1}{n}$$
$$\iff \quad 1/(1 - M') < n \quad \text{(since } 1 - M' > 0\text{)}.$$

By the Archimedean Property, we can take a positive integer n so large that $n > 1/(1 - M')$. Hence, for this value of n, we have

$$1 - \frac{1}{n} > M',$$

so M' is not an upper bound of E_2.

It follows that 1 is the least upper bound of E_2.

(c) The set $E_3 = \{n^2 : n = 1, 2, \ldots\}$ is not bounded above, so it cannot have a least upper bound.

Solution to Exercise D21

(a) We know that 1 is a lower bound of the set
$E_1 = (1, 5]$, since

$$1 \le x, \quad \text{for all } x \in E_1.$$

To show that $m = 1$ is the greatest lower bound
of E_1, we prove that if $m' > 1$, then there is an
element x in E_1 which is less than m'.

Suppose that $m' > 1$. By the Density Property,
there is a real number x such that

$$1 < x < m' \quad \text{and} \quad x \le 5,$$

so $x \in E_1$ and $x < m'$. Thus m' is not a lower
bound of E_1.

Hence 1 is the greatest lower bound of E_1.

(b) We know that 0 is a lower bound of the set
$E_2 = \{1/n^2 : n = 1, 2, \ldots\}$, since

$$0 < 1/n^2, \quad \text{for } n = 1, 2, \ldots.$$

To show that $m = 0$ is the greatest lower bound
of E_2, we prove that if $m' > 0$, then there is an
element $1/n^2$ in E_2 such that $1/n^2 < m'$.

If $m' > 0$, then we have

$$\frac{1}{n^2} < m' \iff n^2 > \frac{1}{m'}$$
$$\iff n > \sqrt{1/m'}.$$

We can take a positive integer n so large that
$n > \sqrt{1/m'}$. Hence the first inequality holds for
this value of n, and so m' is not a lower bound of
E_2.

It follows that 0 is the greatest lower bound of E_2.